Lecture 22: Backpropagation

CS 3780/5780

1 Forward Pass

Forward pass computes the output of each neuron in the neural network given the input and the weights and biases of the network.

1. $z^0 = x$
2. For $t = 1$ to T
   (a) $u^t = W^{[t]} z^{t-1} + b^t$
   (b) $z^t = \sigma(u^t)$
3. End For
4. $\hat{y} = \alpha^\top z^{[T]} + b$

Now once we have done our forward pass, we would have calculated u^t's, z^t's and $\hat{y}$ for a given input x and given some current parameters for the neural network. Forward pass goes layerwise from layer 1 all the way to the lst output layer.

2 Backward Pass

The backward pass is performed after the forward pass, in which we have computed the u^t's, z^t's and $\hat{y}$. The main idea for computing the gradients efficiently is to use the chain rule of calculus. To simplify things first think about the case when each layer has one neuron and input is one dimensional so that we dont need gradients but just derivatives. Note note that

$$u^s = W^{[s]} z^{s-1} + b^t$$

and so

$$\frac{d\ell(\hat{y}, y)}{dW^{[s]}} = \frac{d\ell(\hat{y}, y)}{d\hat{y}} \cdot \frac{d\hat{y}}{du^T} \cdot \frac{du^T}{du^{T-1}} \cdots \frac{du^{s+1}}{du^s} z^{s-1}$$

Hence, if we use the notation

$$\delta^t = \frac{d\ell(\hat{y}, y)}{d\hat{y}} \cdot \frac{d\hat{y}}{du^T} \cdot \frac{du^T}{du^{T-1}} \cdots \frac{du^{t+1}}{du^t}$$

then, since we have that

$$u^t = W^{[t]} \sigma(u^{t-1}) + b^t$$

we have that

$$\frac{du^t}{du^{t-1}} = W^{[t]} \sigma'(u^{t-1})$$

and hence, if we recursively define

$$\delta^{t-1} = (W^{[t]} \sigma'(u^{t-1})) \delta^t$$

starting with $\delta^T = \frac{d\ell(\hat{y},y)}{d\hat{y}}\alpha z^T$ That is, δ^t's can be computed using a backward pass starting from last layer (output layer) working back towards the first input layer. Hence we can conclude that:

$$\frac{d\ell(\hat{y},y)}{dW^{[s]}} = \delta^s z^{s-1}$$

and similarly we can conclude that,

$$\frac{d\ell(\hat{y},y)}{db^s} = \delta^s$$

In the general multilayer, multi-neuron (higher dimensional) case, we get the backward pass shown in the slides.