Answer Key — Decision Theory First Exam

I

1. This choice function violates Sen's α as $b \in C(\{a,b,c\})$ and $b \notin C(\{a,b\})$.
2. Since the von Neumann-Morgenstern axioms are staisfied there is an expected utility representation with some utility function u. As 2 for sure is strictly preferred to the probability p' giving 1 with probability 1/2 and 2 with probability 1/2 we have $u(2) > 1/2u(1) + 1/2u(2)$. This implies that $u(2) > u(1)$. We are asked to show that the probability p giving 2 with probability 3/4 and 3 with probability 1/4 is strictly preferred to the probability q giving 1 with probability 1/4, 2 with probability 1/2 and 3 with probability 1/4. This is equivalent to showing that

   $$\frac{3}{4}u(2) + \frac{1}{4}u(3) > \frac{1}{4}u(1) + \frac{1}{2}u(2) + \frac{1}{4}u(3)$$

   Algebra shows that this is equivalent to $u(2) > 1/3u(1) + 2/3u(2)$, which is equivalent to $u(2) > u(1)$.

   You could also show this using the independence axiom with $\alpha = \frac{1}{2}$ and $r = (0, \frac{1}{2}, \frac{1}{2})$.
3. $v(x) = (u(x))^3$ represents the same preferences as does u because v is a strictly increasing transformation of u.
4. (a) $x \succ y$ is $x \succeq y$ and $y \not\succeq x$. For the relation in this question this is $x > y + \frac{1}{2}$. To show transitivity we need to show that if $x \succ y$ and $y \succ z$ then $x \succ z$. Note that $x \succ y$ and $y \succ z$ imply $x > y + \frac{1}{2}$ and $y > z + \frac{1}{2}$.This implies that $x > z + 1$. So $\succ$ is transitive.

   (b) $y \sim x$ is $y \succeq x$ and $x \succeq y$. For the relation in this question this is $y \in [x - \frac{1}{2}, x + \frac{1}{2}]$. To see that $\sim$ is not transitive let $y = x + \frac{3}{8}$ and $z = y + \frac{3}{8}$ for some real number x. Then $y \sim x$ and $z \sim y$. But $z = x + \frac{3}{4}$, so $z \notin [x - \frac{1}{2}, x + \frac{1}{2}]$. Thus $z \not\sim x$.

II

- Show that for any $A, B \in P(X)$, $x \in C(A \cup B, \succ)$ implies $x \in C\Big(C(A, \succ) \cup C(B, \succ), \succ\Big)$.

  Since $x \in C(A \cup B, \succ)$ we have for all $y \in A \cup B$, $y \not\succ x$. Then as $x \in A \cup B$, $x \in c(A, \succ)$, $x \in c(B, \succ)$ or both. So $x \in c(A, \succ$

$) \cup c(B, \succ)$. As $c(A, \succ) \cup c(B, \succ) \subset A \cup B$ we have for all $y \in c(A, \succ) \cup c(B, \succ)$, $y \not\succ x$. Thus, $x \in C\Big(C(A, \succ) \cup C(B, \succ), \succ\Big)$.

- Show that for any $A, B \in P(X)$,$x \in C\Big(C(A, \succ) \cup C(B, \succ), \succ\Big)$ implies
  $x \in C(A \cup B, \succ)$.
  We know that $c(A, \succ) \neq \emptyset$ and $c(B, \succ) \neq \emptyset$. Let $a \in C(A, \succ)$ and $b \in C(B, \succ)$. Then for all $y \in A$, $y \not\succ a$ and for all $z \in B$, $z \not\succ b$. From $x \in C\Big(C(A, \succ) \cup C(B, \succ), \succ\Big)$, $a \in C(A, \succ)$ and $b \in C(B, \succ)$ we have $a \not\succ x$ and $b \not\succ x$. Then by Negative Transitivity we have for all $y \in A$, $y \not\succ x$ and for all $z \in B$, $z \not\succ x$. Thus $x \in C(A \cup B, \succ)$.

Alternative proofs using Sen's axioms are also possible.

III Most people did pretty well on parts (a), (b), and (c), and pretty badly on parts (d) and (e). A typical grade on this problem was 20–25 (out of 40). Very few people got over 30.

(a) [8 points] The retailer has to choose how many sneakers to buy and how many boots to buy (buying N altogether). Thus, there are $N+1$ possible acts, $\{a_0, \ldots, a_N\}$, where act a_i denotes buying i boots (and $N-i$ sneakers). The uncertainty involves how many boots he can sell before March. So again, a state can be characterized by a number in $\{0, \ldots, N\}$. To get 8 points, you had to make clear exactly what the acts and states were.

You didn't have to say the following, although it's useful for later: If he performs the act a_i and is in state s, then he will sell $\min(i, s)$ boots before March (he can can sell up to s boots, but can't sell more than he buys), he'll sell all the leftover, namely $i - \min(i, s)$, after March, and sell all s sneakers. He makes a \$6 profit on all the boots he sells before March, he loses \$2 on all the bootshe sells after March, and makes \$3 on all the sneakers he sells. Thus, the outcome of perfoming act a_i in state s is that he makes a profit of $\$[6\min(i, s) - 2(i - \min(i, s)) + 3(N - i)$.

(b) [6 points] If he buys i boots, in the worst case, the demand for boots before March will be 0 (and he'll lose \$2 on each pair of boots he buys). The maximin act a_0: buying only sneakers.

(c) [6 points] Since the retailer makes more of a profit on boots sold before March then on sneakers sold, in the best case, he buys N boots and sells them all before March. Thus, the maximax act is a_N: buying only boots.

(d) [10 points] Intuitively, if you (the retailer) perform act a_i (i.e., buys i boots), then there are two things that could cause you regret: (i) you sell fewer boots than you buy (in which case you wish you had bought fewer boots) or (ii) the demand for boots was greater than i (in which case you wish you had bought more boots). What will cause you the most regret is that the demand for boots is 0 or N. Which of these two cases will cause more regret depends on how many boots you buy. For example, if you only buy 2 boots, then if the demand is 0, that won't cause you much regret (you only bought two boots anyway), but you will suffer serious regret if you discover the demand was actually N. Similarly, if you buy $N-2$ boots, then if the demand is N, you won't suffer a lot of regret, but you will suffer a lot of regret if the demand is 0. This should help you see that you are unlikely to minimize the maximum regret by buying either 0 or N boots. You need to hedge your bets. Here are the technical details.

Clearly the best act in state s (where s boots would be bought before March) is a_s, buying exactly s boots. If the retailer performs act a_i, in state $s \leq i$, the retailer's regret is $5(i-s)$: compared to the best act in this state, namely if $i \geq s$, his regret is $\$5(i-s)$: he loses \$2 on each of the $i-s$ boots that are sold after March, and he could have made \$3 had he bought sneakers instead of those boots. If $i < s$, then the regret is $\$3(s-i)$; he could have bought $s-i$ more boots and sold them before March, making \$3 more profit on each one. Thus, the maximum regret for act a_i is $\max(5i, 3(N-i))$. Intuitively, if he buys i boots and i is high, he feels the most regret ($5i$) if he doesn't sell any boots before March. If he buys i boots and i is low, then his regret is highest $3(N-i)$ if he could have sold N boots, had he only bought them. As i increases, $5i$ increases and $3(N-i)$ decreases. The retailer minimizes his maximum regret for that i at which $5i = 3(N-i)$, which happens when $i = 3N/8$. [Although you didn't have to say this, here's the intuition in more detail. If the retailer buys $3N/8$ boots, the regret is $15N/8$ (just plug $3N/8$ in for i in the expression for regret). If $i < 3N/8$, the retailer should have bought more boots: the regret is maximized when the demand for boots is N, and $3(N - i) > 15N/8$. If $i > 3N/8$, the retailer should have bought fewer boots: regret is maximized when the demand for boots is 0, and $5i > 15N/8$.]

If you only considered the acts a_0 and a_N, you typically got 3/10. It's not enough just to consider these cases.

(e) [10 points] The analysis here is similar in spirit to that for part (d). Again, the best act in state s is a_s; it gives revenue $6s + 3(N - s) = 3s + 3N$. The multiplicative regret of act a_i in state s is thus $(6\min(i, s) - 2(i - \min(i, s)) + 3(N - i)))/(3s + 3N)$. In particular, if $i \leq s$, the regret is $(3i + 3N)/(3s + 3N)$, and if $i > s$, the regret is $(4s - 5i + 3N)/(3s + 3N)$. The first expression is clearly minimized (i.e., gives the worst case regret) when $s = N$; the regret in this case is $(-5i + 3N)/6N$. The second expression is minimized $s = 0$. (The easiest way to see this is to observe that the second expression is equivalent to $4/3 - ((5i + N)/(3s + N)$. You could also use calculus or just claim that it was true by experimentation. We weren't fussy.) This gives a regret of $(3N - 5i)/3N = (6N - 10i)/6N$. Thus, the worst case regret of act a_i is $\min(6i + 3N, 6N - 10i)/6N$. The act a_i with the best worst-case regret is the one for which $6i + 3N = 6N - 10i$; i.e. $i = 3N/16$.