Decision Theory I
Problem Set 3

1. Consider a finite set of prizes X and probabilities P on them. Suppose that an individual's preferences $\succ$ on P have an expected utility representation with utility function $u : X \to \Re$. Show that $\succ$ satisfies the independence axiom.

2. Consider a finite set of prizes X and probabilities P on them. Suppose that an expected utility maximizer's preferences $\succ$ on P have an expected utility representation with utility function $u : X \to \Re$. Show that $v : X \to \Re$ also represents $\succ$ if and only if there exist real numbers $a > 0$ and b such that $v(\cdot) = au(\cdot) + b$.

3. An expected utility maximizing individual with wealth w will lose $a < w$ with probability p, and will have no loss with probability $1 - p$. Assume that the individual's utility function u satisfies $u''(m) < 0 < u'(m)$ for all $m > 0$. She is offered insurance at a premium of r per unit. One unit of insurance pays 1 in the event of a loss and 0 otherwise. If she buys x units of insurance her wealth will be $w - a - rx + x$ if there is a loss and $w - rx$ if there is no loss. The insurance is"actuarialy fair", so $r = p$. How many units of insurance will she purchase?

4. Evaluate the following argument: Max, who is a risk averse expected utility maximizer, is offered an opportunity to buy fair insurance. (The premium equals the expected payoff as in the problem above.) He decides not to buy the insurance. His reasoning is that buying insurance is really gambling; you pay a premium and you may not get a payoff. Fair insurance is a fair gamble, but because he is risk averse he would not accept a fair gamble and thus should not accept fair insurance.

5. Suppose that an expected utility maximizing individual's utility function, $u(\cdot)$, for for wealth satisfies $u'(m) > 0 > u''(m)$ for all $m > 0$. The individual's initial wealth is $w > 0$. She is offered a bet with probability p of winning t and probability $1 - p$ of losing t. Assume that $0 < p < 1$ and $0 < t < w$.

   a. Assume that $p > \frac{1}{2}$. Show that if t is small enough then she will accept the bet.

b. Assume that $p = \frac{1}{2}$. Is there any value of t (with $0 < t < w$) for which she will accept the bet?

6. GRAD An expected utility maximizing investor must decide how to allocate his initial wealth $w > 0$ between money and a risky asset. The gross rate of return on money, m, is 1. The gross rate of return on investment in the risky asset, x, is $1 + H > 1$ with probability p and $1 + L < 1$ with probability $1 - p$. So if the investor's portfolio is described by (m, x) his random wealth is $w + Hx$ with probability p and $w + Lx$ with probability $1 - p$. The individual's utility function is u with $u'(w) > 0 > u''(w)$ for all $w > 0$. His choice of x is constrained to lie in $[0, w]$.

   (a) Suppose that the expected net rate of return on the asset, $pH + (1 - p)L$, is strictly positive. Prove that the individual will hold a strictly positive amount of the risky asset.

   (b) Now suppose that $u(w) = \log(w)$. Assume that the optimal amount of the risky asset satisfies $w > x* > 0$. Show that if H increases then so does the optimal amount of the risky asset $x*$.