Decision Theory I
Problem Set 1

1. Show that if $\succ$ is negatively transitive and asymmetric then $\succ$ is transitive.

2. Suppose that $\succ$ is a partial, asymmetric and transitive relation on a finite set X. Let $c(\cdot, \succ)$ be the choice function induced by $\succ$. Does this choice function necessarily satisfy Sen's β?

3. Suppose $X = \{x, y, z\}$. Consider a choice function such that $C(\{x, y\}) = \{x\}$, $C(\{x, z\}) = \{z\}$ and $C(\{y, z\}) = \{y\}$. Does this choice function satisfy Sen's α and β?

4. The set of alternatives is $X = \{a, b, c\}$ and $\succ$ is a binary order on X reflecting strict preference. Suppose that for $x \in \{b, c\}$, $x \not\succ a$ and $a \not\succ x$. Suppose also that $b \succ c$. Can this relation be a strict preference relation? Explain.

   If we want to include the possibility that there is an alternative a that is not comparable to either b or c in our analysis then we would want the condition above on a to be satisfied. What does this example say about non-comparability?

5. Let $\succ$ be a binary relation on a finite set X. Define $\succeq$ by: $x \succeq y$ if $y \not\succ x$. Show

   (a) If $\succeq$ is complete then $\succ$ is asymmetric.

   (b) If $\succeq$ is transitive then $\succ$ is negatively transitive.

6. We left the proof that a revealed preference relation defined from a choice function satisfying Sen's α and β is negatively transitive unfinished. Finish it.

   Here is where we were in the proof: Suppose that for some $x, y, z \in X$ we have $y \not\succ x$ and $z \not\succ y$. We need to show that $z \not\succ x$. Showing that $x \in C(\{x, y, z\})$ is sufficient. Suppose that $x \notin C(\{x, y, z\})$. Then $y \in C(\{x, y, z\})$, $z \in C(\{x, y, z\})$ or both. We showed that if $y \in C(\{x, y, z\})$ then $x \in C(\{x, y, z\})$. You need to show that if $z \in C(\{x, y, z\})$ then $x \in C(\{x, y, z\})$.

7. A binary relation that is reflexive, symmetric and transitive is called an equivalence relation. An equivalence relation partitions a set into equivalence classes. Suppose that $\succ$ is a strict preference relation on X. Then by Proposition 2.4 of Kreps we know that $\sim$ is an equivalence relation on X. For each $x \in X$ define its equivalence class by $I(x) = \{y \in X | y \sim x\}$. Show that the sets $I(x)$ partition X and that the sets $I(x)$ are strictly ranked as Professor Halpern drew them on the board in class on Thursday. (A collection of sets $\{A_1, \ldots, A_N\}$ partitions X if each $x \in X$ is in at least one A_i and $A_i \cap A_j = \emptyset$ for all $i \neq j$. The equivalence classes are strictly ranked if, for all $x, y \in X$, (1) if $I(x) \neq I(y)$, then either $x \succ y$ or $y \succ x$ and (2) if $x \succ y$, then $x' \succ y'$ for all $x' \in I(x)$ and $y' \in I(y)$.)

8. **GRAD:** In the statement of Sen's α and β we allow the sets A and B to be any subsets of X. So when we proved that these axioms imply a rational revealed preference relation we allowed ourselves to use information about choices from arbitrary subsets of X. Find the smallest class of subsets of X such that if we require α and β to be satisfied on this class of sets the claim in the revealed preference theorem is true. More precisely, find a set B of subsets of X such that if a choice function c is defined on the sets in B, and satisfies α and β on these sets, then there is a preference order $\succ$ on X such that $c(\cdot, \succ)$ agrees with c on B.

9. **GRAD:** In class in the proof of the revealed preference theorem we defined strict revealed preference. Weak revealed preference is defined as follows: $x \succeq y$ if $x \in C(\{x, y\})$. Define induced strict revealed preference $\succ^*$ from revealed preference $\succeq$ by: $x \succ^* y$ if $x \succeq y$ and $y \not\succeq x$. Suppose the choice function satisfies Sen's α and β. Are strict revealed preference and induced strict revealed preference the same relation?