Decision Theory II: Problem Set 1

1. This exercise examines comonotonic independence. Assume that the state space $S = \{s_1, \ldots, s_n\}$ is a finite state space.

   (a) Show that if f and g are comonotonic random variables, then there exists a permutation $t_1, \ldots, t_n$ of the states in S such that $f(t_1) \leq \ldots \leq f(t_n)$ and $g(t_1) \leq \ldots \leq g(t_n)$.

   (b) Show (using part (a)) that if ν is a nonadditive probability and f and g are comonotonic random variables, then $E_\nu(f + g) = E_\nu(f) + E_\nu(g)$, where E_ν denotes Choquet expectation. (Note that for a probability measure μ, $E_\mu(f + g) = E_\mu(f) + E_\mu(g)$ for all f and g. This exercise shows that additivity holds for comonotonic functions using Choquet expectation, for an arbitrary nonadditive probability.)

   (c) Show that if $\alpha > 0$ and ν is a nonadditive probability, then $E_\nu(\alpha f) = \alpha E_\nu(f)$. Show by means of a counterexample that this is not in general true if $\alpha < 0$. (Note that for probabilistic expectation, this property holds for all α.)

   (d) Recall that, given a utility function u on prizes, we can associate with each horse lottery h a random variable u_h on S, where $u_h(s) = \sum_{z \in Z} h(s)(z)u(z)$. (This makes sense since $h(s)$ is a probability distribution on Z, the set of prizes, with finite support.) Show that comonotonic independence holds; that is, if we define $f \succ g$ iff $E_\nu(u_f) > E_\nu(u_g)$, show that if (i) f and h are comonotonic, (ii) g and h are comonotonic, and (iii) $f \succ g$, then for all $\alpha \in [0, 1]$, we have that $\alpha f + (1 - \alpha)h \succ \alpha g + (1 - \alpha)h$. (Recall that f and g are comonotonic if $f(s) \succ f(t)$ iff $g(s) \succ g(t)$. $f(s)$ is the constant act that always returns the lottery $f(s)$. Thus, comonotonicity means that $u_f(s) > u_f(t)$ iff $u_g(s) > u_g(t)$.)

2. Show that A1 and A4$'$ together imply A4.

3. This exercise shows that A1 and A4 imply A4$'$. So assume that $\succ$ is an order on acts for which A1 and A4 hold. If p is a lottery over prizes, let $\overline{p}$ denote the constant function on states that always gives p. Recall that fXg is the act that agrees with f on X and with g on X^c.

   (a) Suppose that $S_1 \supseteq S_2$, $T_1 \supseteq T_2$, and both $S_1 - S_2$ and $T_1 - T_2$ are nonnull. Show that $\overline{p}(S_1\overline{q}) \succ \overline{p}S_2\overline{q}$ iff $\overline{p}T_1\overline{q} \succ \overline{p}T_2\overline{q}$. (Hint: show that $\overline{p}S_1\overline{q} \succ \overline{p}(S_1 - \{s\})\overline{q}$ iff $\overline{p}(S_1 \cup \{s'\})\overline{q} \succ \overline{p}S_1\overline{q}$, if $s \in S_1$, $s' \notin S_1$, and s, s' are both nonnull.)

   (b) Let f be an arbitrary act. Show that $f(S - \{s\})\overline{p} \succ f(S - \{s\})\overline{q}$ iff $\overline{p} \succ \overline{p}(S - \{s\})\overline{q}$. (Hint: again change things one state at a time.)

   (c) Now prove A4$'$, using (a) and (b).

4. Show that convexity implies the covering property. (This shows that a convex nonadditive probability must be a lower probability.)

5. The infinite horizon optimality principle is that the value function V^* is the unique solution of $WV = V$ where W is defined by

$$(WV)(s) = \max_a [u(a,s) + \beta \sum_{s'} V(s')P(a,s)(s')].$$

   for each $s \in S$. In class we proved for the finite action and state case that V^* is a solution. Using the same assumptions as in class prove that the solution is unique.

6. We discussed a procedure to use the operator W defined above to approximate the value function V^*. This procedure is to apply W repeatedly to any guess V (with $||V|| \leq C/(1-\beta)$) about V^*. The claim that we made was that

$$||W^T V - V^*|| \leq \beta^T 2C/(1-\beta)$$

   where C is the bound on the reward function u. Prove this claim. (You can use the fact that V^* solves the optimality equation.)