CS**5643**

17 Stability and stiffness

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Simple example

Suppose we are simulating a particle in 2D on the x axis

- $\dot{x} = -x$ —decays exponentially towards y axis
- as part of larger simulation, might get pulled off axis, so install strong force to keep it there
- $\dot{y} = -ky$ —decays towards x axis very fast (to avoid drift)

Integrate this system in time

- write down forward Euler step
- when is it stable?

Timestepping that same system with backward Euler

- write down backward Euler step
- when is it stable? … always!

Simple example on a slant

Suppose the actual problem is not axis aligned

- direction **u** = [1 1] is the “soft” axis, **v** = [1 -1] is the “stiff” one
- write this down as a linear ODE with a 2x2 matrix
- if we know the vectors **u** and **v** it is simple to analyze

forward Euler

- fairly obvious we have the same behavior as before

backward Euler

- timestep equation involves solving a system
- knowing the secret basis, we can rewrite matrix and see same behavior again

More general case

The first generalization of this game is a symmetric positive definite matrix

- the secret vectors are the eigenvectors, an orthonormal basis
- the damper stiffnesses are the eigenvalues
- the time step constraint involves the largest eigenvalue
- still only exponential decay type solutions

Full generalization is to a positive definite matrix

- similar analysis applies
- time step constraint asks for all eigenvalues in a certain region
- backwards Euler automatically puts all eigenvlaues in that region.