CS**5643**

16 Elastic rod simulation

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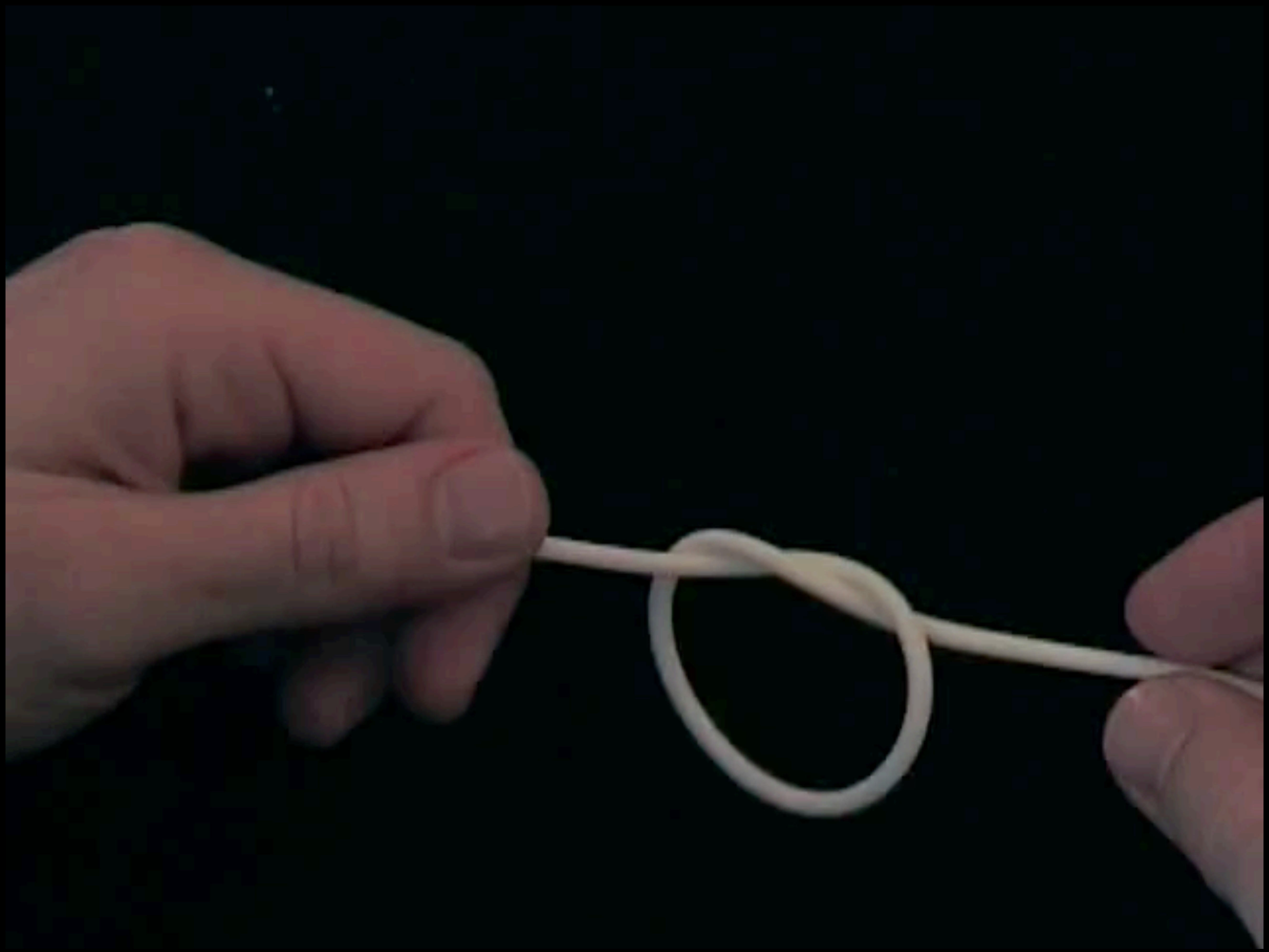
Cornell University
Spring 2025



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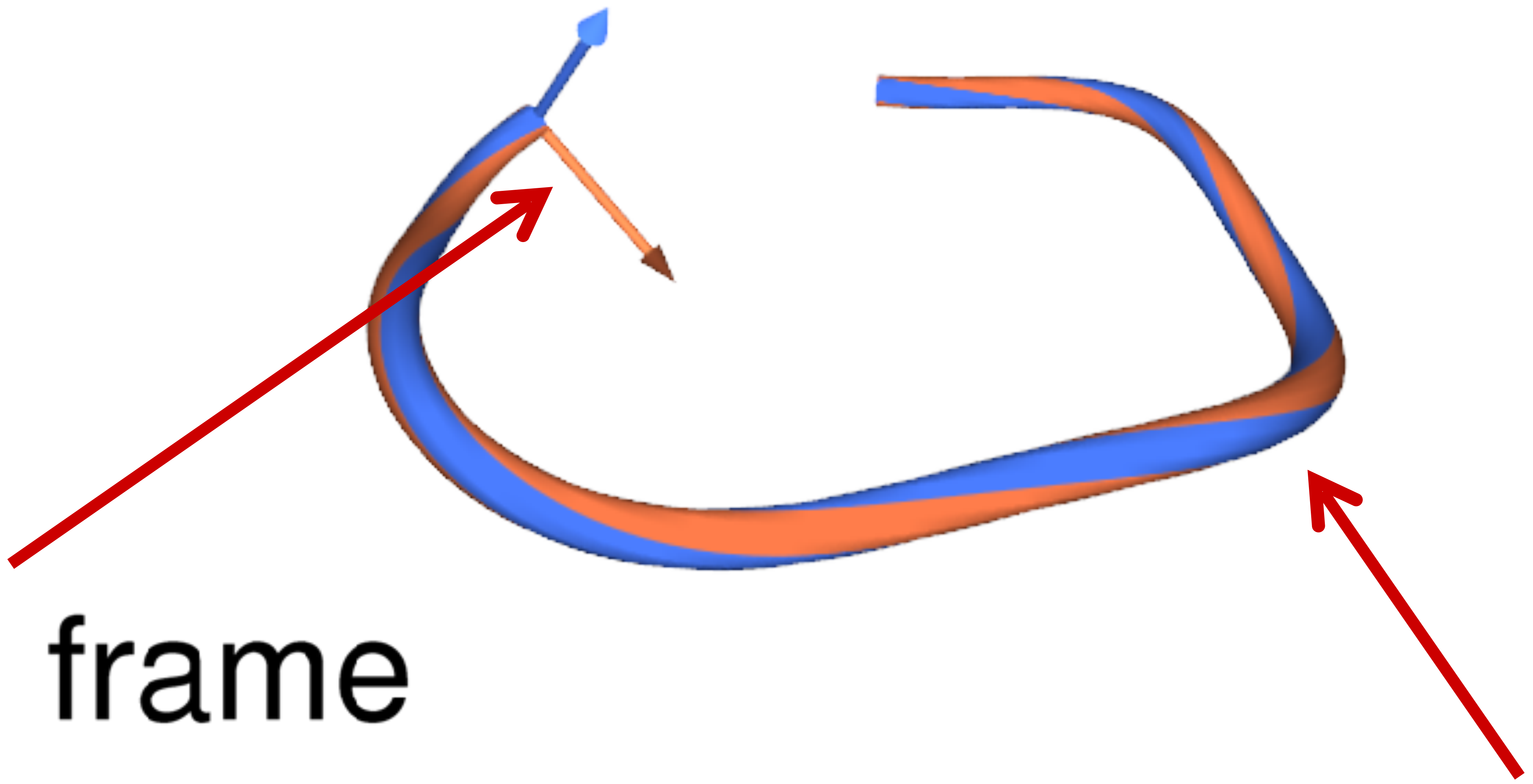
Kirchhoff Rod Model



adapted frame

$\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{t} = \boldsymbol{\gamma}'(s)\}$

curve $\boldsymbol{\gamma}(s)$

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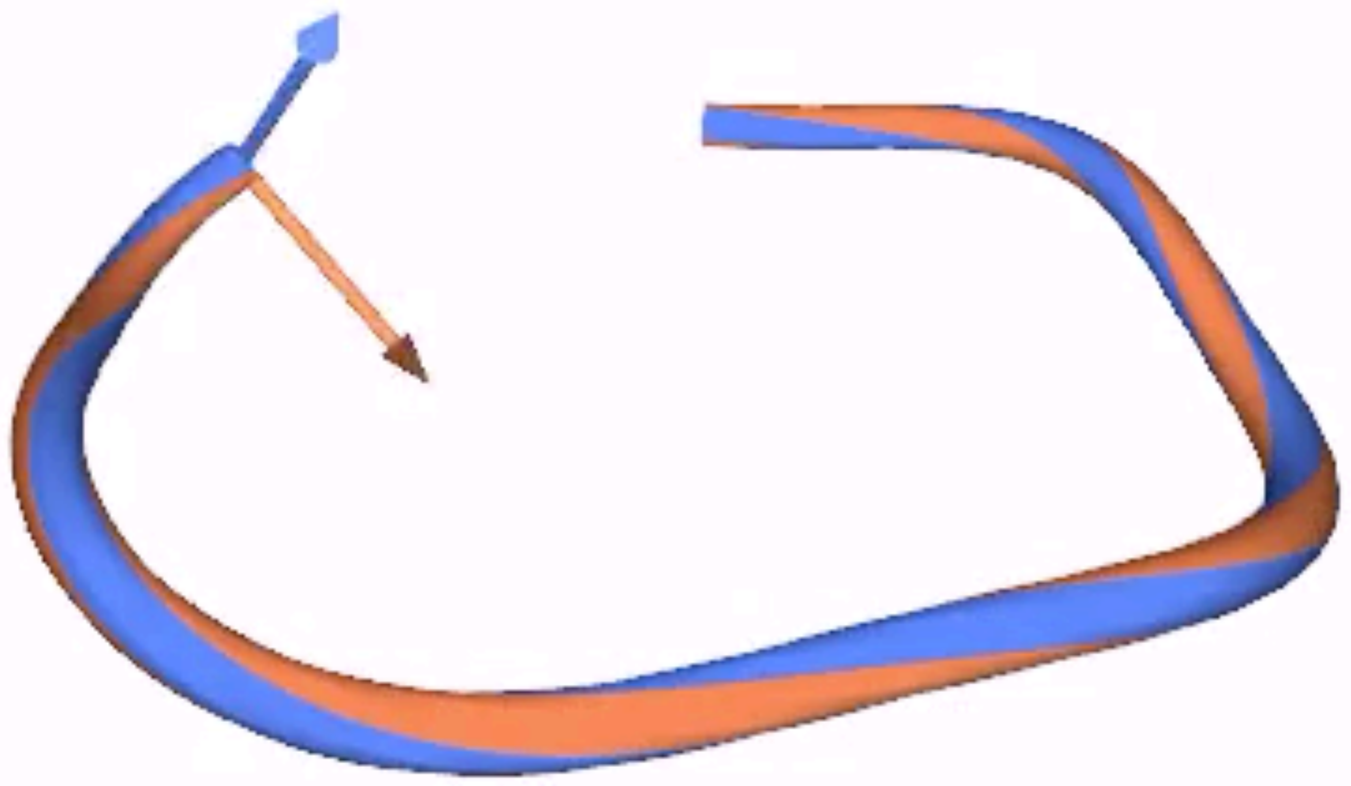
Kirchhoff Rod Model



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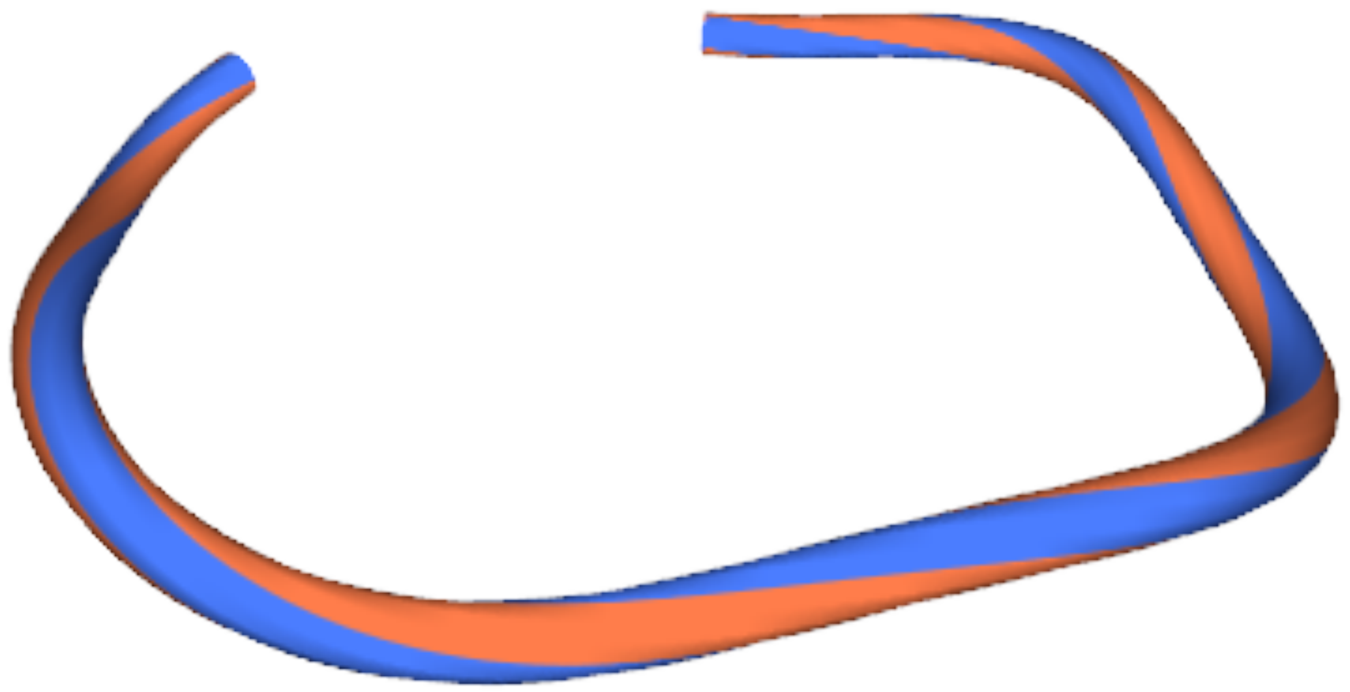
Energy of an Elastic Rod



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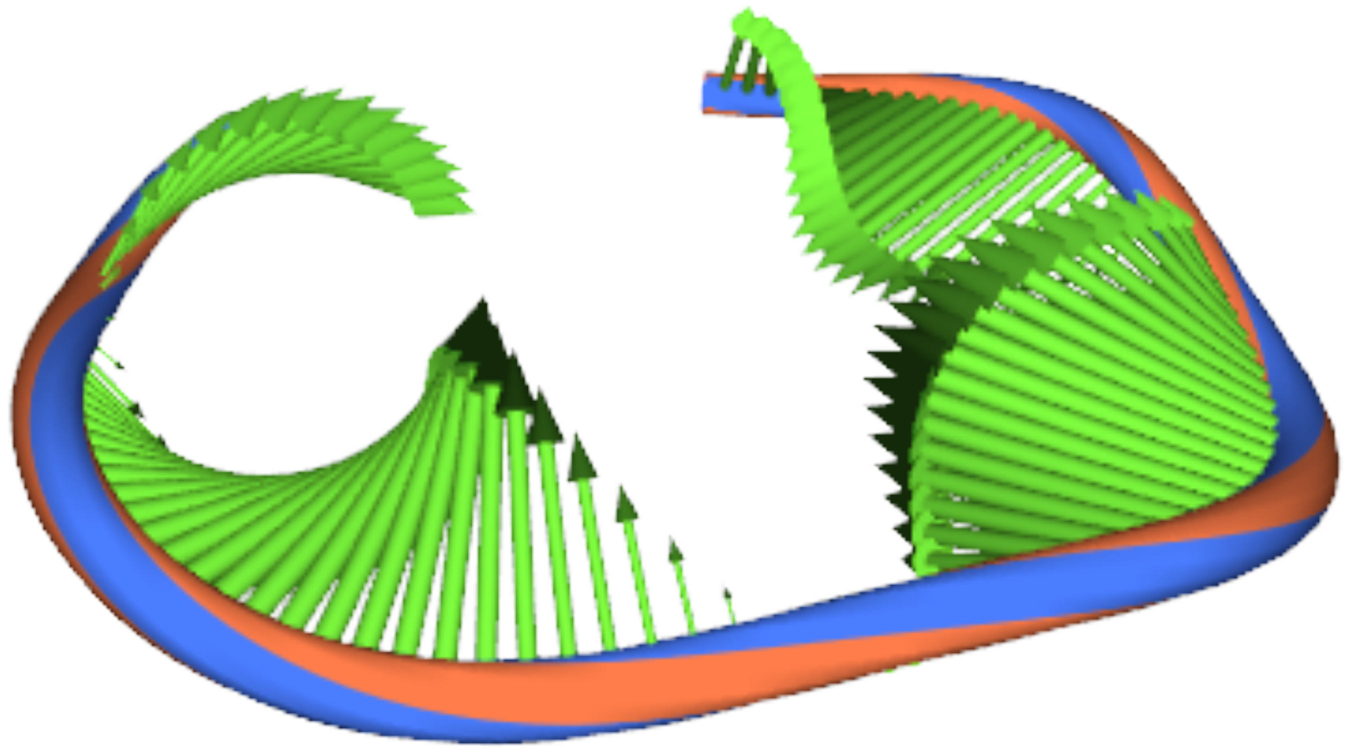
Energy of an Elastic Rod

curvature
$\boldsymbol{\kappa} = \boldsymbol{\gamma}''$



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Energy of an Elastic Rod

curvature
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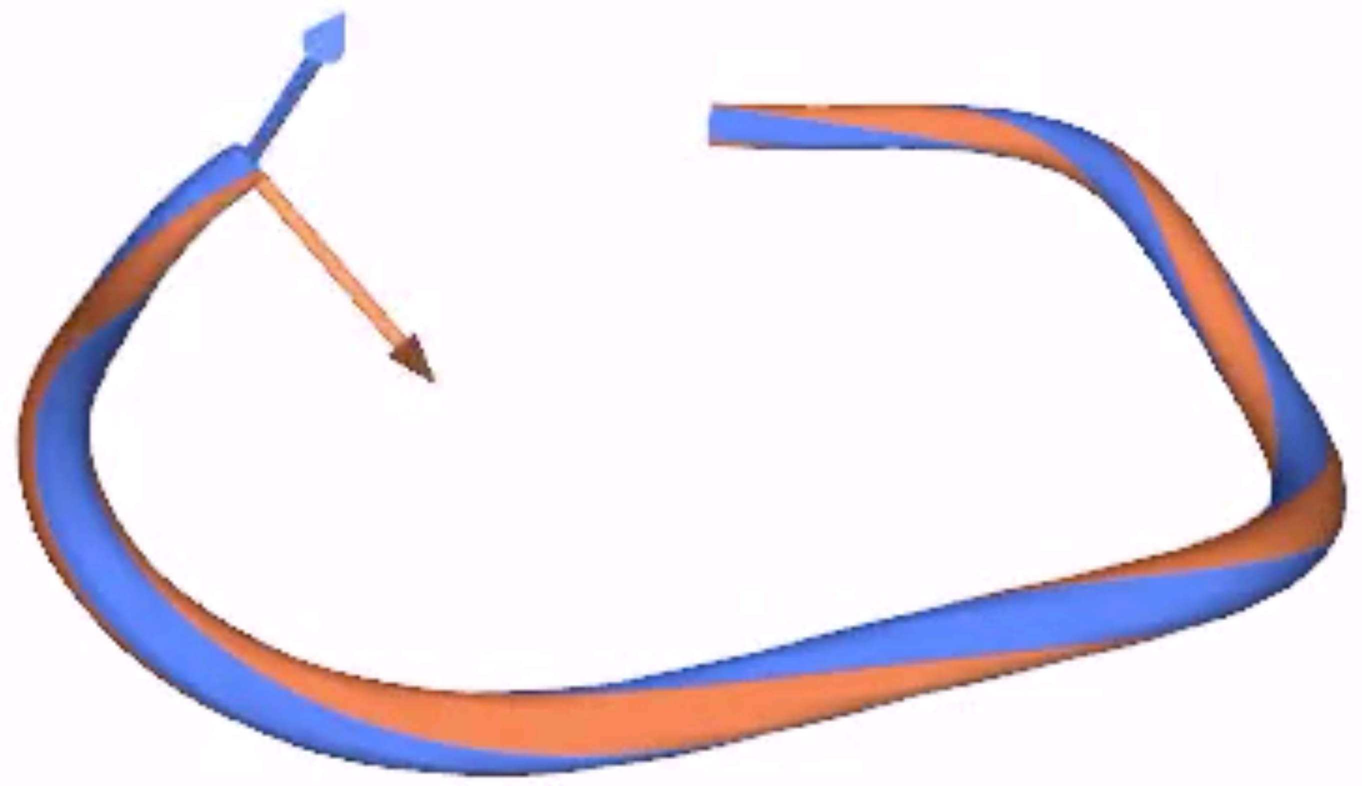

twist
$\mathbf{m_1}' \cdot \mathbf{m_2}$

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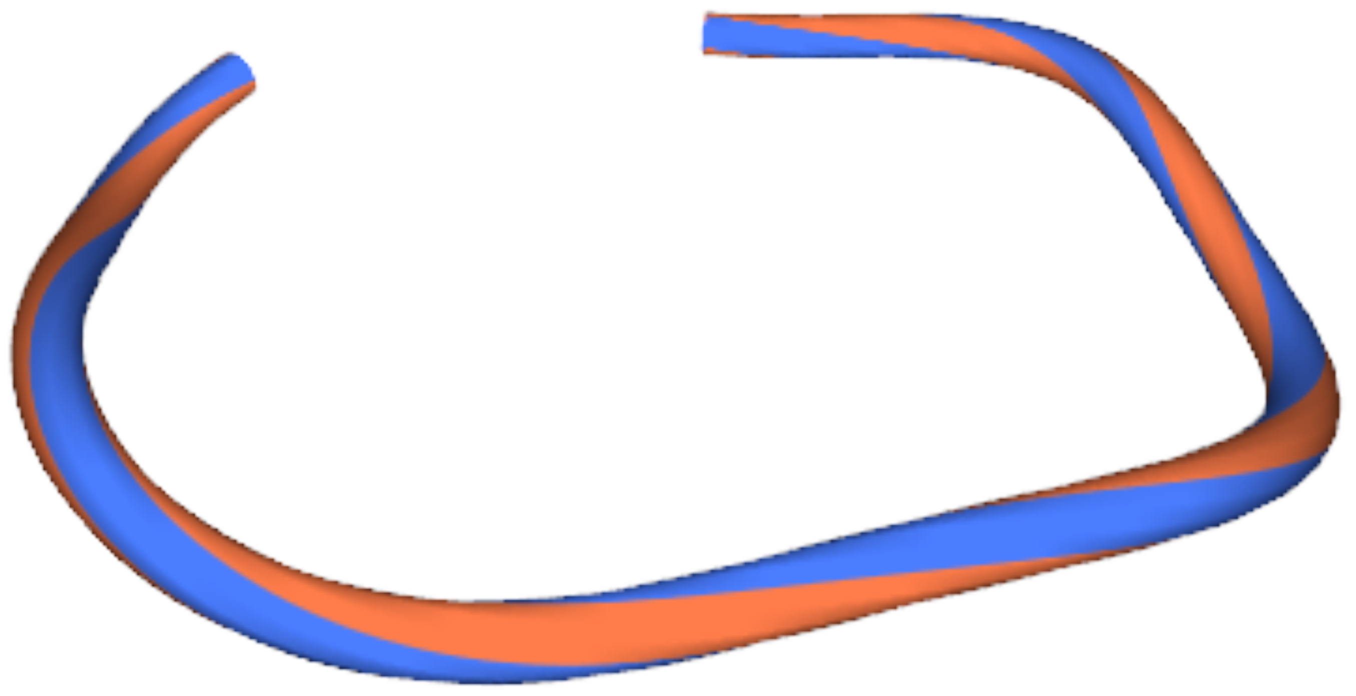
Energy of an Elastic Rod



curvature
$\boldsymbol{\kappa} = \boldsymbol{\gamma}''$

twist
$\mathbf{m_1}' \cdot \mathbf{m_2}$

$$\frac{1}{2}\int \alpha \boldsymbol{\kappa}^2 \mathrm{d}s + \frac{1}{2}\int \beta m^2 \mathrm{d}s$$

bending twisting

(slide by Miklós Bergou from *Discrete Elastic Rods* presentation)

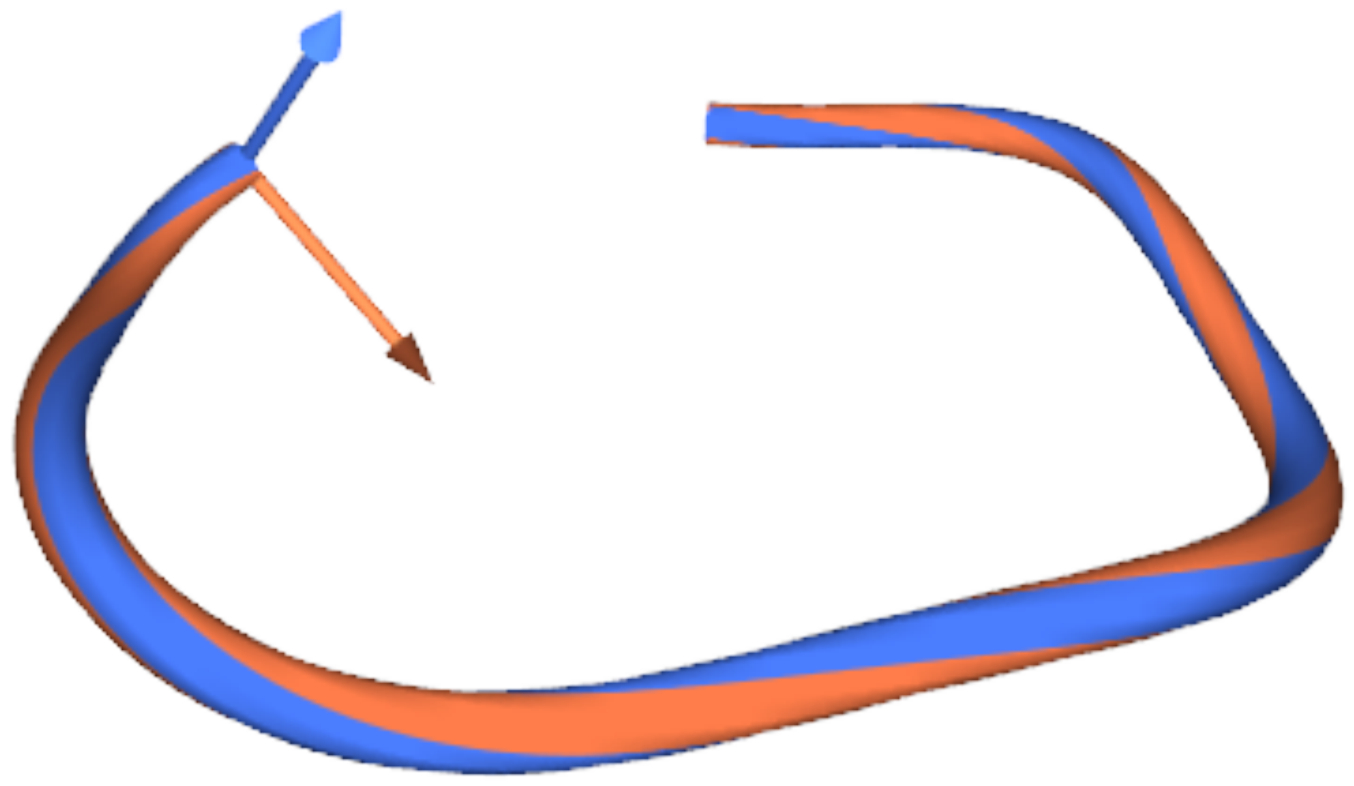
Degrees of Freedom

Represent configuration

- centerline
- adapted material frame

Everything else follows

- energy
- forces
- time stepping



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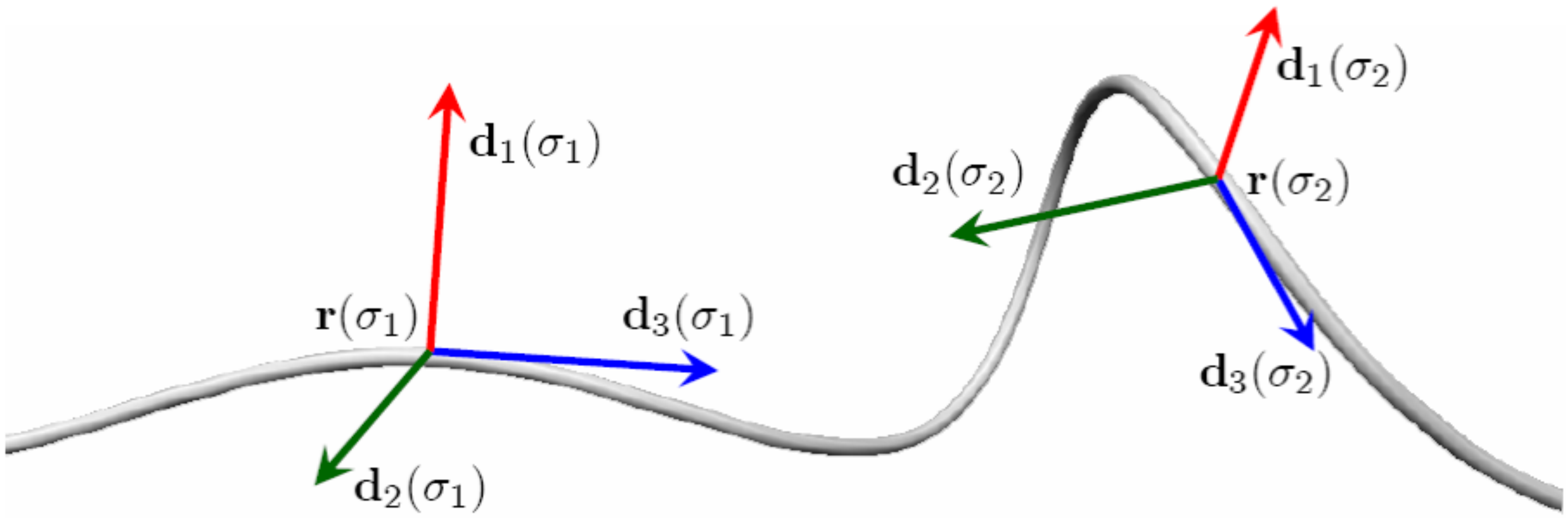
CORDE

Example of maximal parameterization

- represent vertices explicitly as points (3 vars per vertex)
- represent frames explicitly as quaternions (4 vars per segment)
- tie vertices and frames together using constraints
  - 1 constraint per segment for quaternion length
  - 3 constraints per segment for alignment and edge length
- end: 7N + 3 variables - 4N constraints = 3N + 3 DoF

CORDE



[Spillman & Teschner 2007]

Super-helices

Example of minimal parameterization

- write down one vertex, one frame at the end (6 vars)
- write down only the curvatures (3 vars / segment) — minimal, no constraints!
- total: 3N + 6 vars, 3N + 6 DoF

Formulating the simulator is a bit more complex

- Lagrangian mechanics to get equations of motion from
  - elastic potential energy: super simple, just square the DoFs
  - gravitational potential: a bit complex since every segment affects the position of all following
  - kinetic energy: yet more complex but still doable!

Super-helices

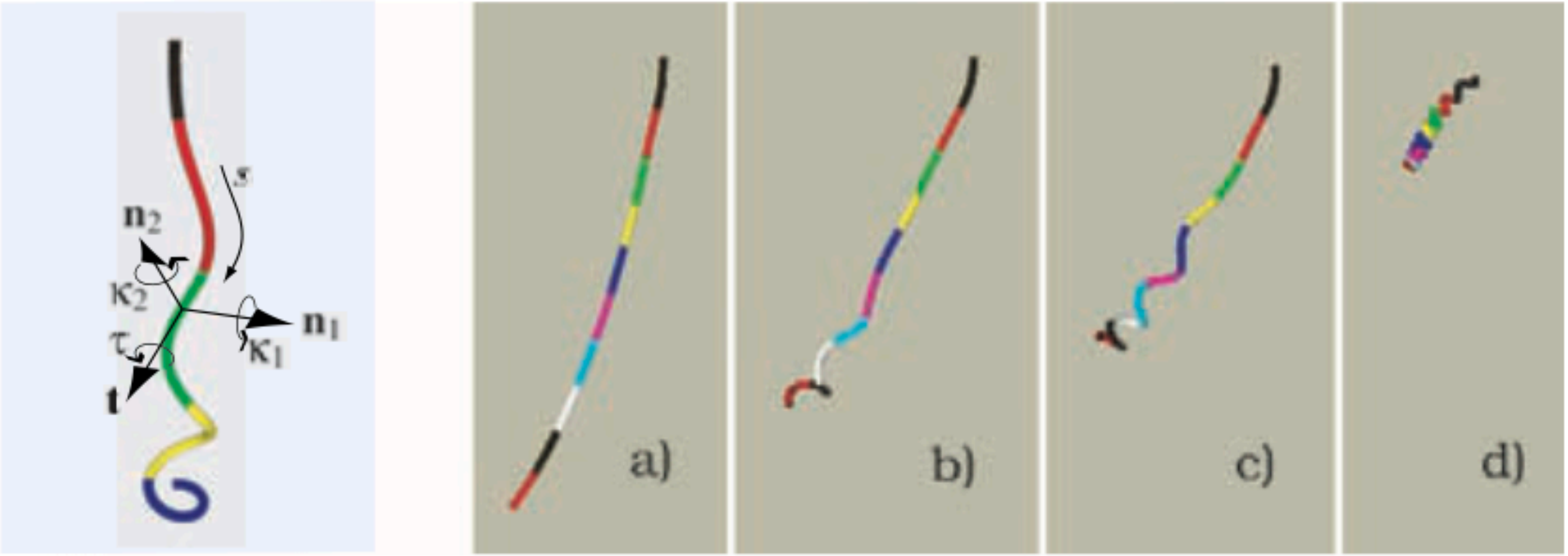


Figure 3: Left, geometry of Super-Helix. Right, animating Super-Helices with different natural curvatures and twist: a) straight, b) wavy, c) curly, d) strongly curly. In this example, each Super-Helix is composed of 10 helical elements.

[Bertails et al. 2006]

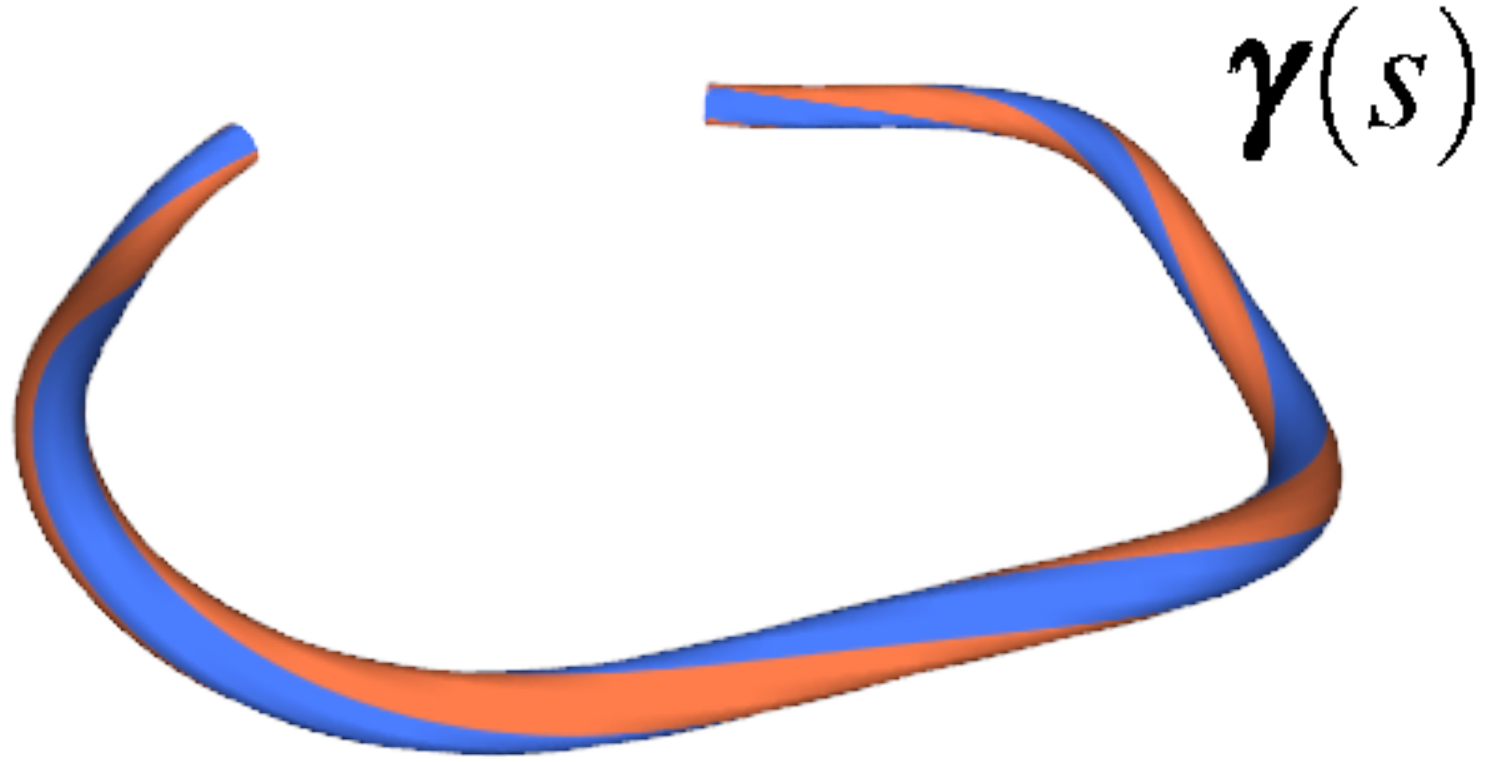
Discrete Elastic Rods

Compromise "happy medium" arrangement

- represent vertices explicitly as points — 3 vars / vertex
- represent frames using just a twist angle — 1 var / segment, minimal!
- measure twist angle relative to a well-defined reference frame
- still need length constraints (1 constraint per segment)
- total: 4N + 3 variables, N constraints = 3N + 3 DoFs

Curve + Angle Representation

Explicit centerline



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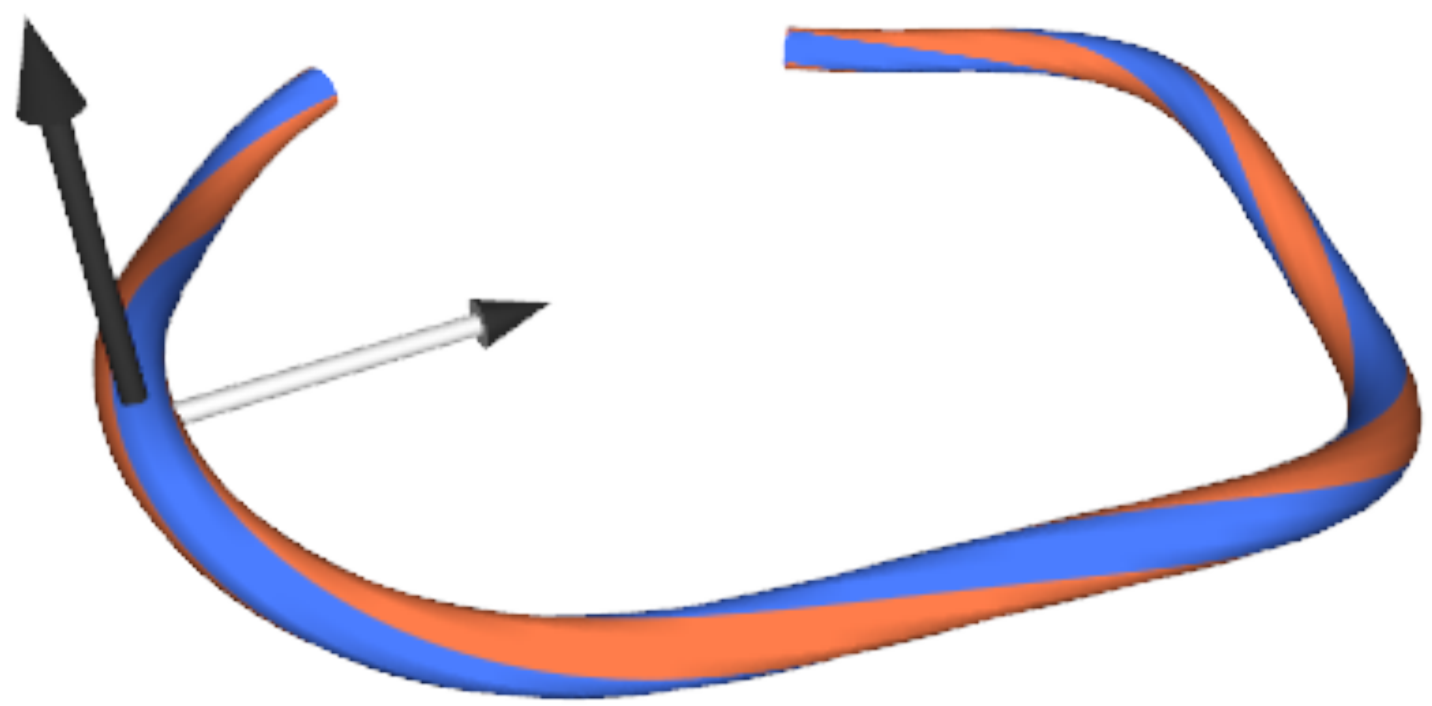
Curve + Angle Representation

Explicit centerline

Reference frame



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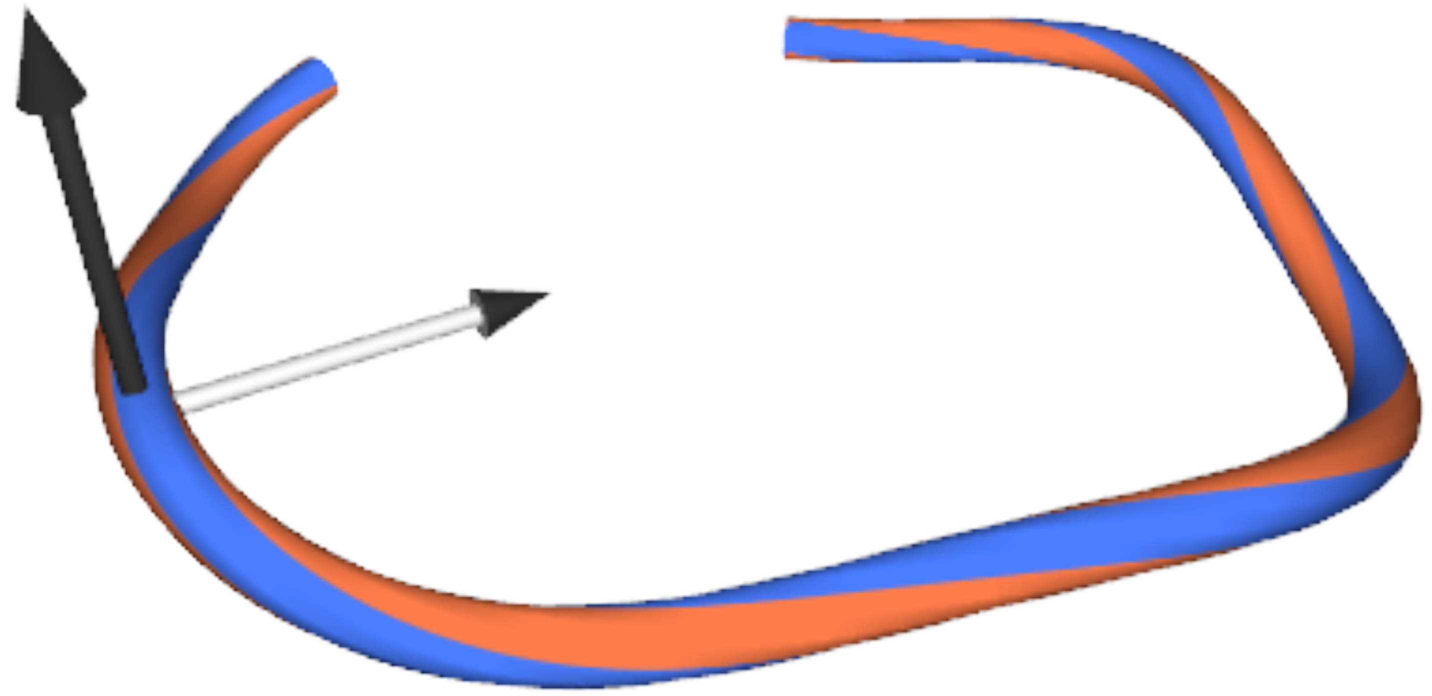
Curve + Angle Representation

Explicit centerline

Reference frame

- $\{\mathbf{u}, \mathbf{v}\}$ twist-free



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Curve + Angle Representation

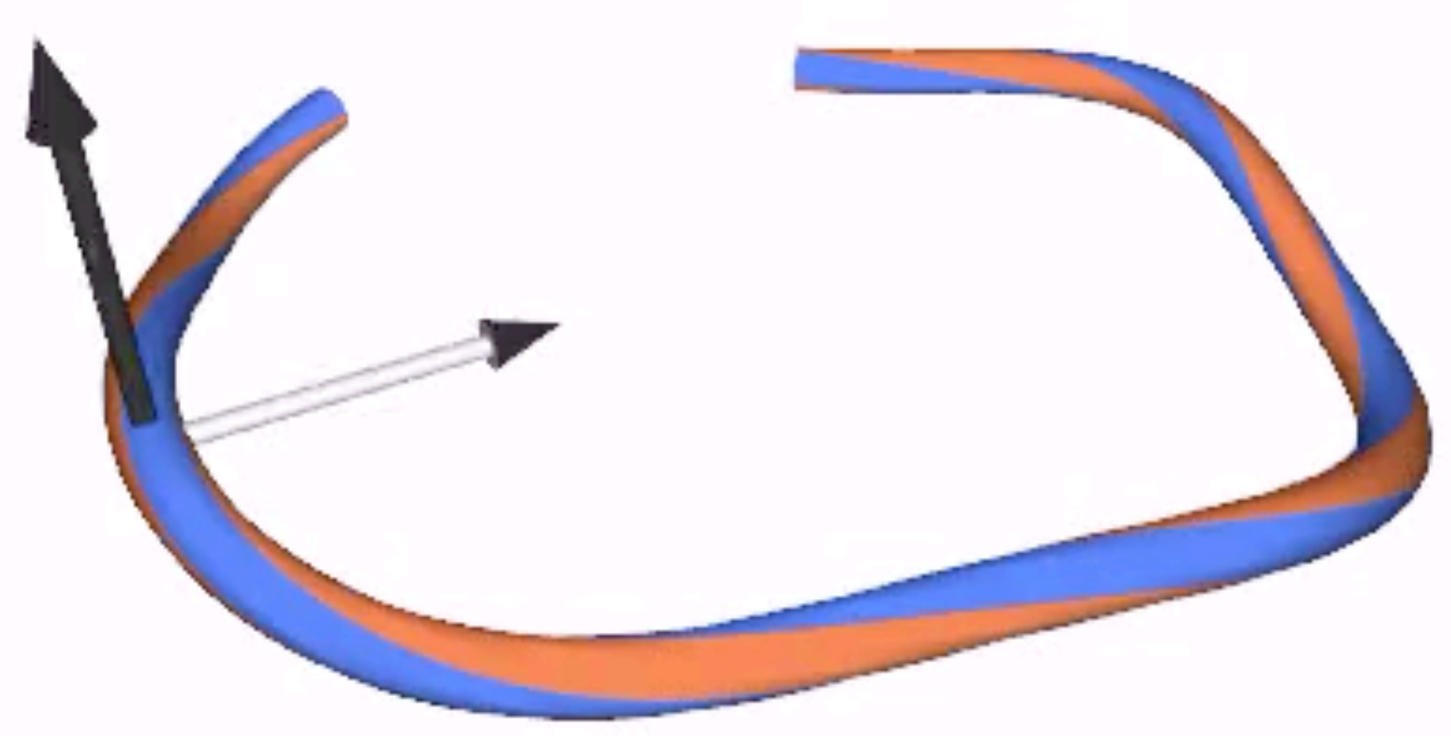
Explicit centerline

Reference frame

- $\{\mathbf{u}, \mathbf{v}\}$ twist-free

Material frame

- relative angle



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Curve + Angle Representation

Explicit centerline

Reference frame

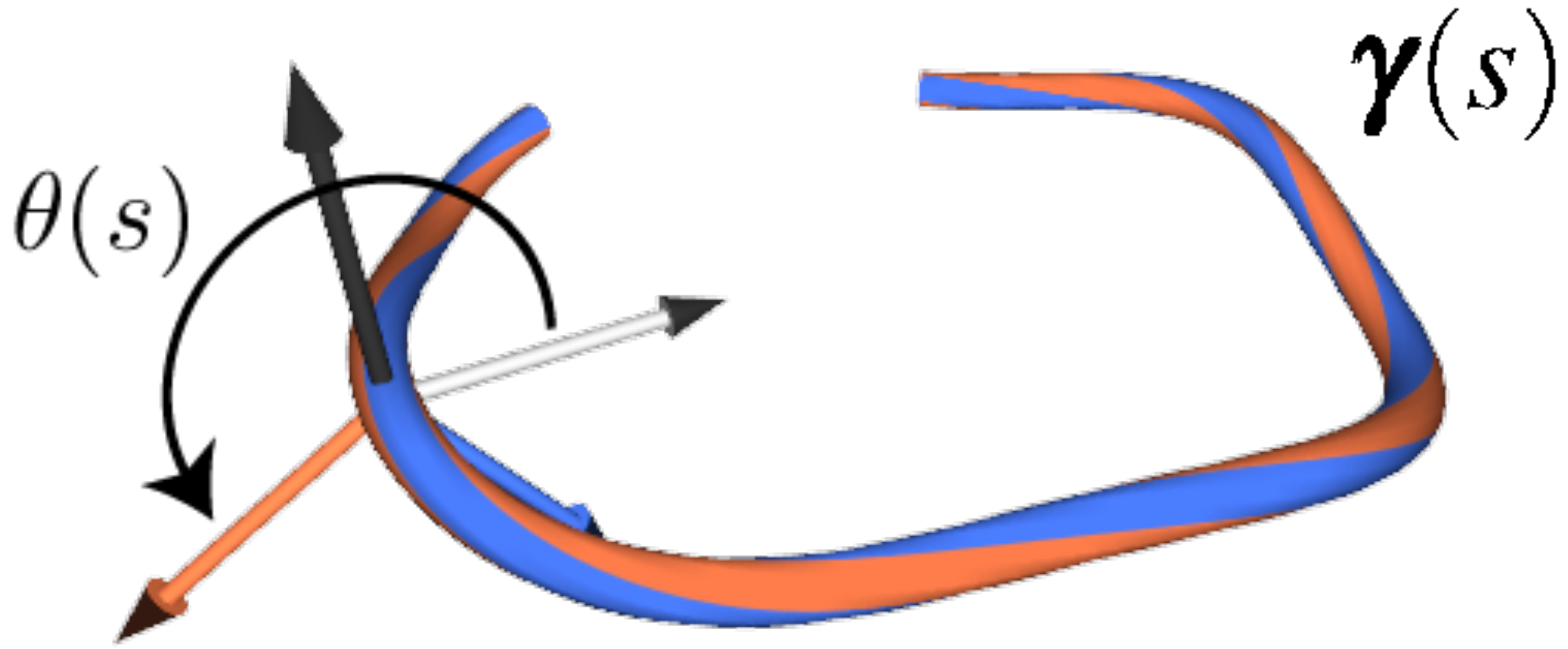
- $\{\mathbf{u}, \mathbf{v}\}$ twist-free

Material frame

- relative angle

Coordinates

- $\boldsymbol{\gamma}(s) : \mathbb{R} \to \mathbb{R}^3$
- $\boldsymbol{\theta}(s) : \mathbb{R} \to \mathbb{R}$



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Curve + Angle Representation

Explicit centerline

Reference frame

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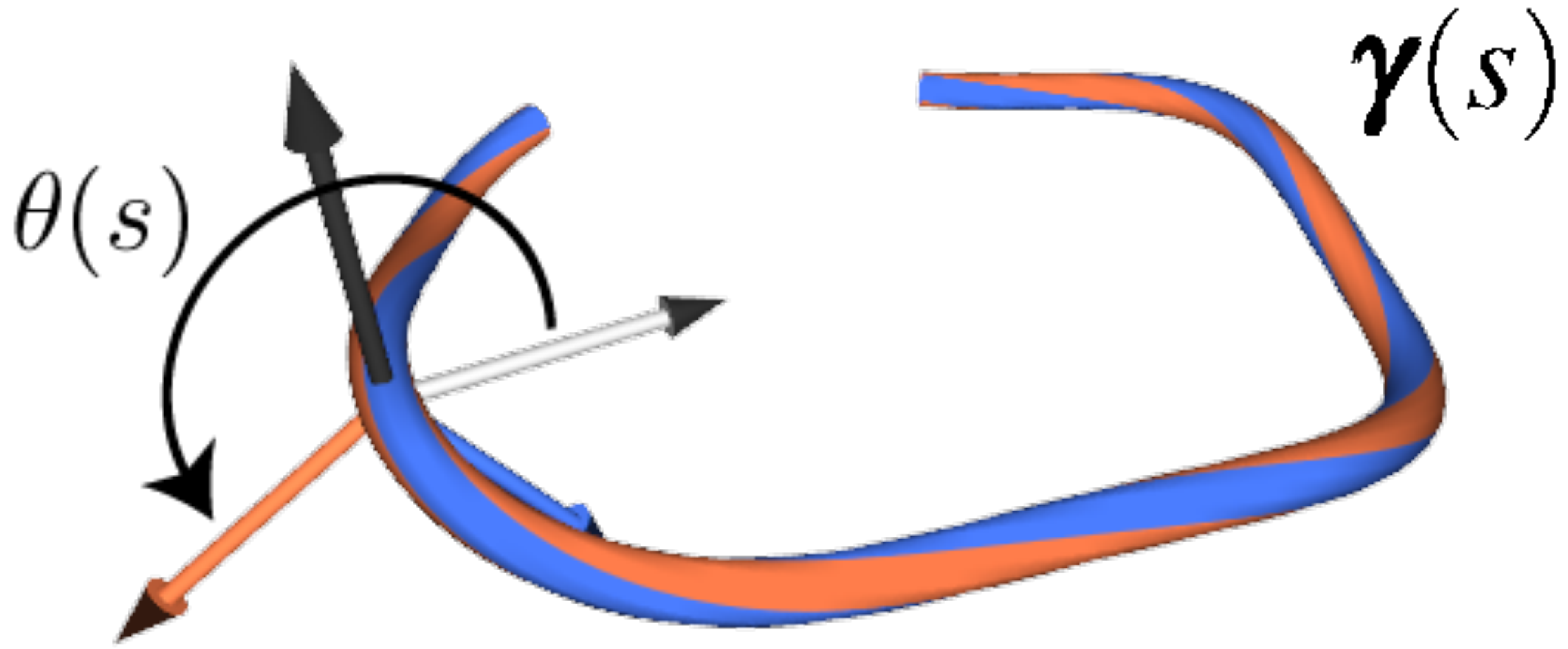

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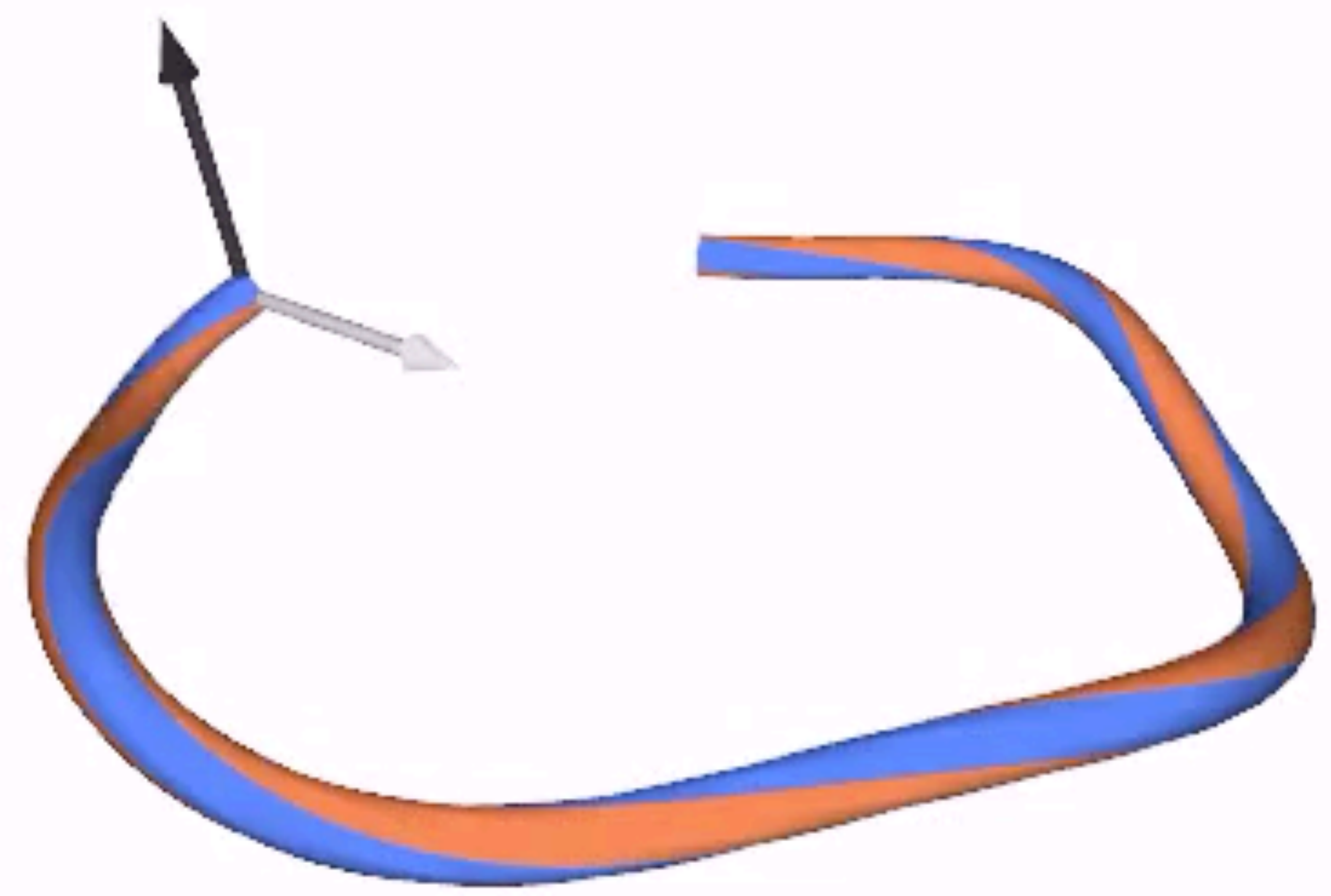
Twist-Free Frame



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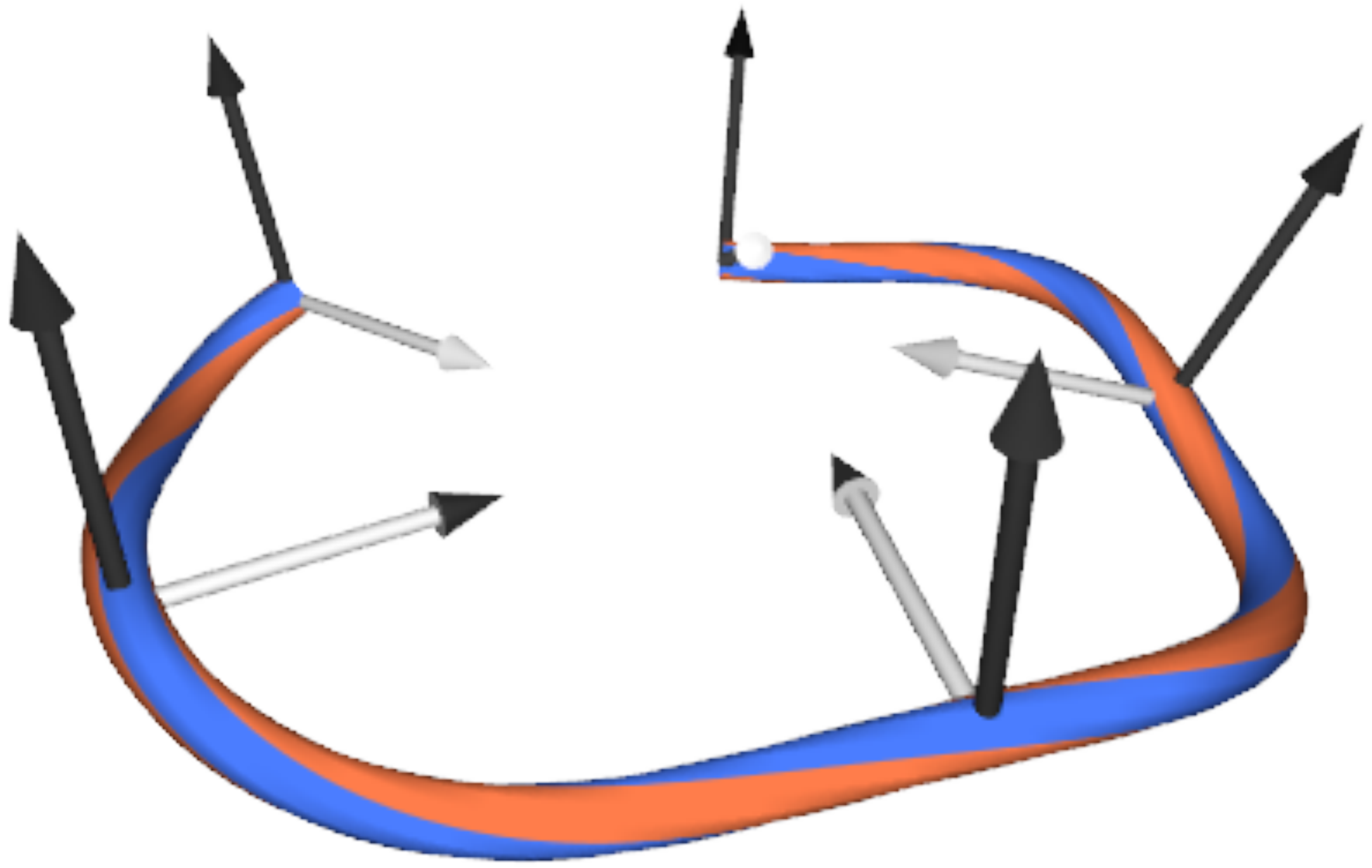
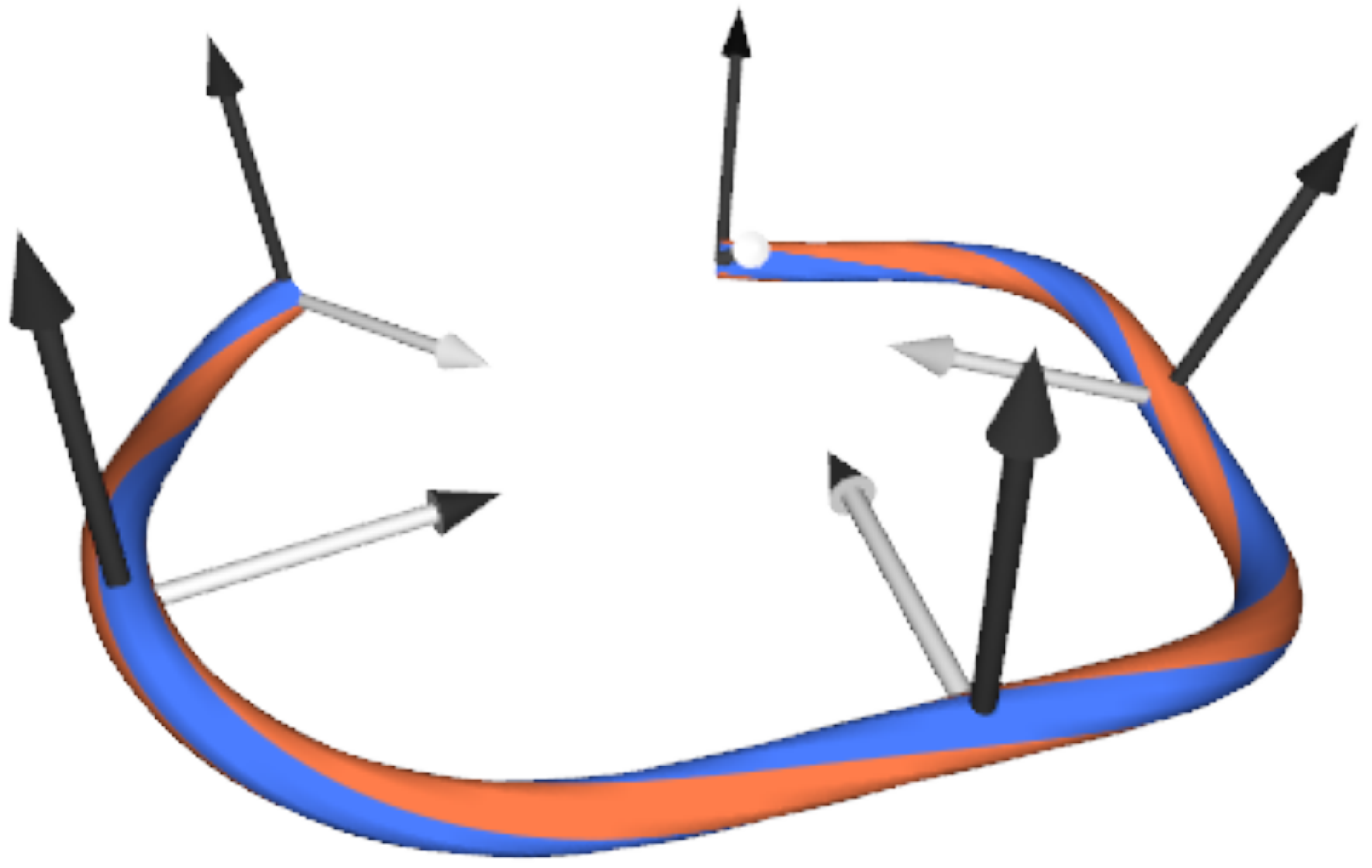
Twist-Free Frame



Requirements
1. no twist about tangent
2. frame must stay adapted

(slide by Miklós Bergou from *Discrete Elastic Rods* presentation)

Parallel Transport



Requirements

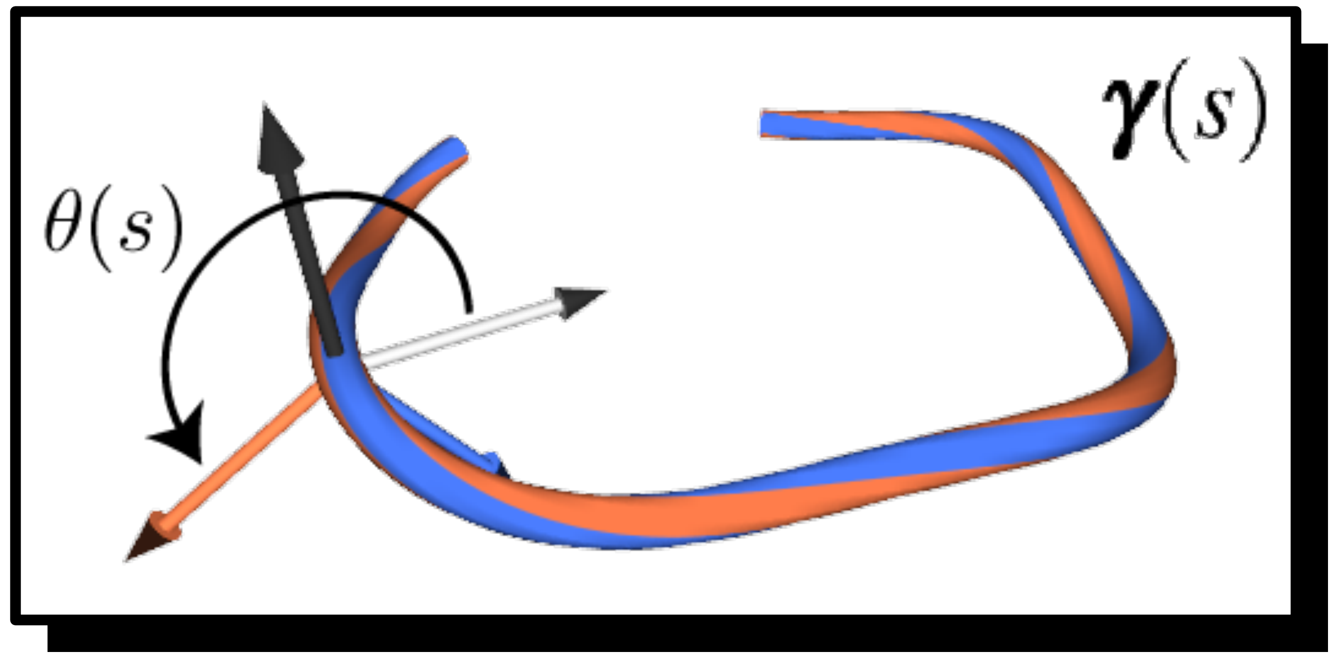
1. no twist about tangent
2. frame must stay adapted

Rotation about binormal

$$\mathbf{u}' = \kappa\mathbf{b} \times \mathbf{u}$$
$$\mathbf{v}' = \kappa\mathbf{b} \times \mathbf{v}$$

(slide by Miklós Bergou from *Discrete Elastic Rods* presentation)

Continuous Setting



How do bending & twisting interact?

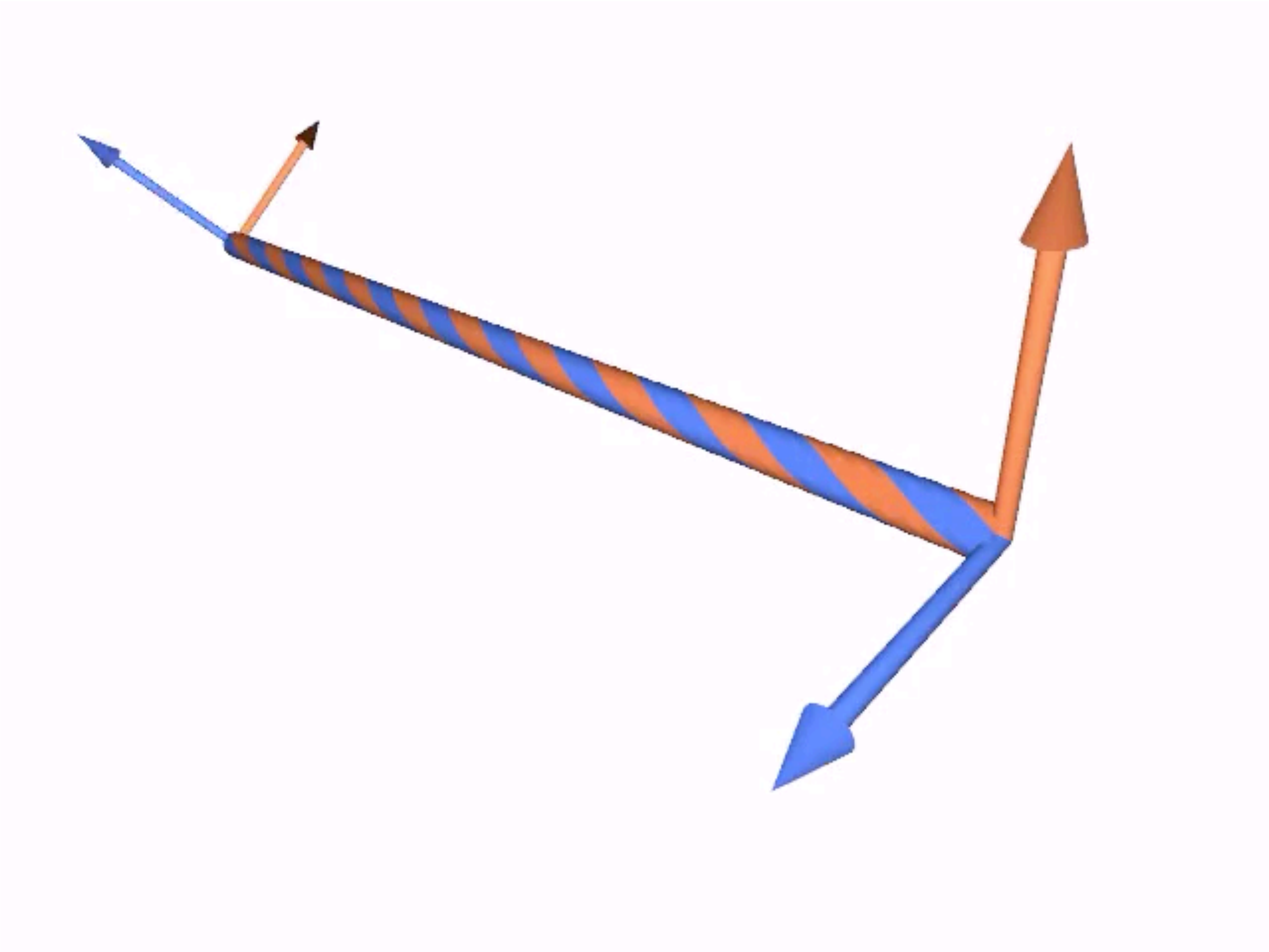
bending   twisting

$$\frac{1}{2}\int \alpha \boldsymbol{\kappa}^2 \mathrm{d}s + \frac{1}{2}\int \beta m^2 \mathrm{d}s$$

$$\boldsymbol{\kappa} = \boldsymbol{\gamma}'' \qquad m = \theta'$$

(slide by Miklós Bergou from *Discrete Elastic Rods* presentation)

Bend & Twist Interaction



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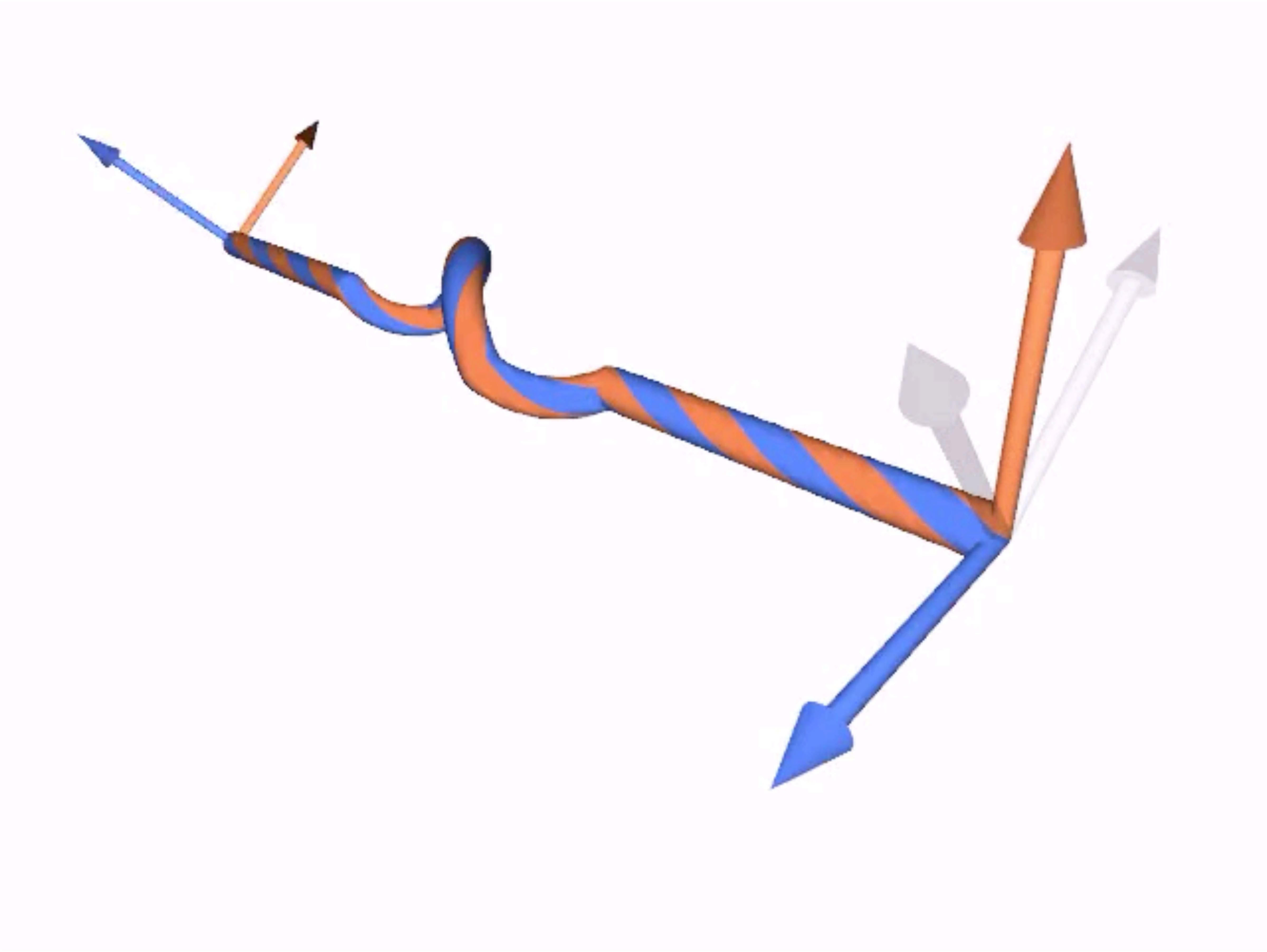
Bend & Twist Interaction



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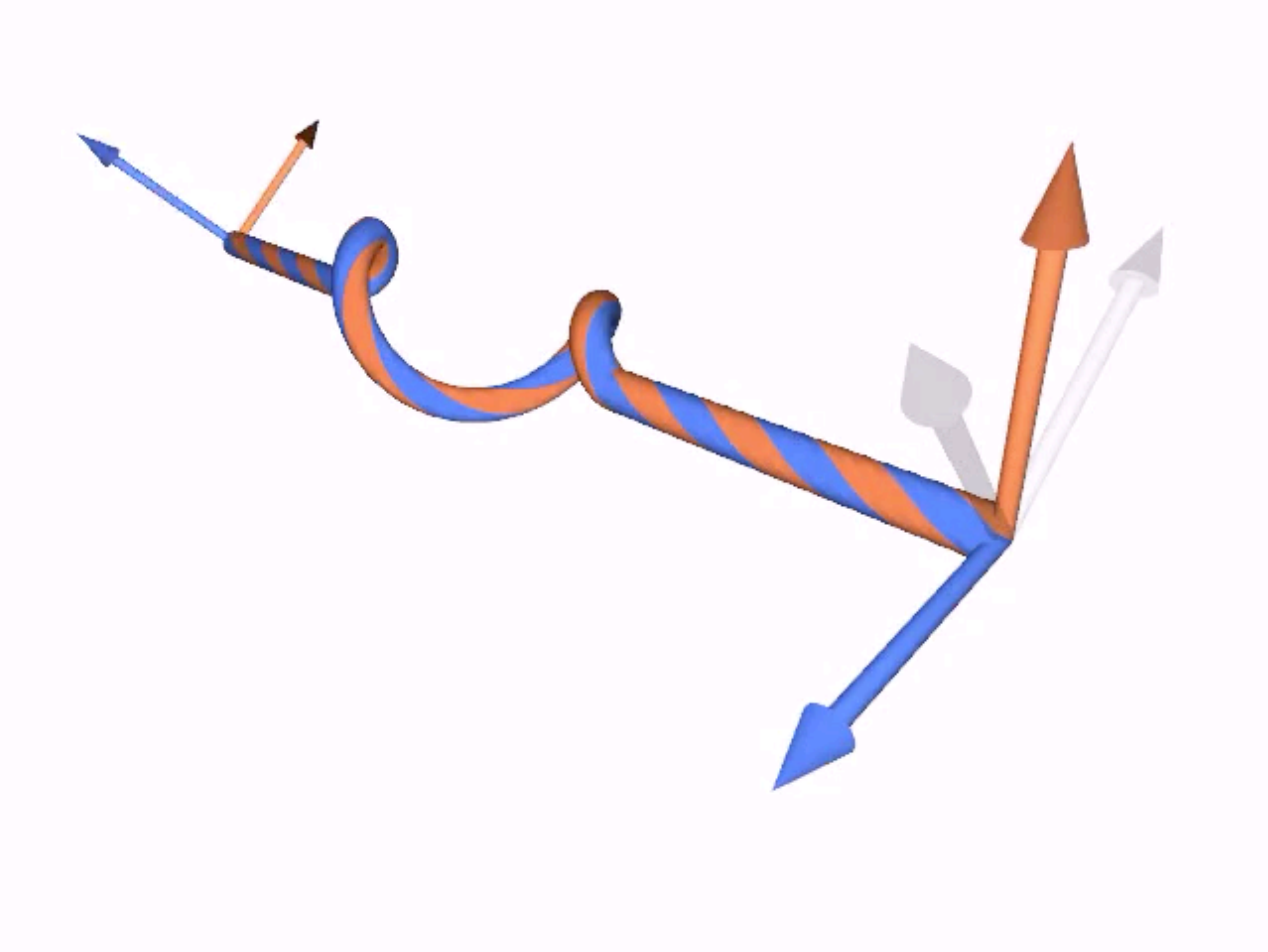
Bend & Twist Interaction



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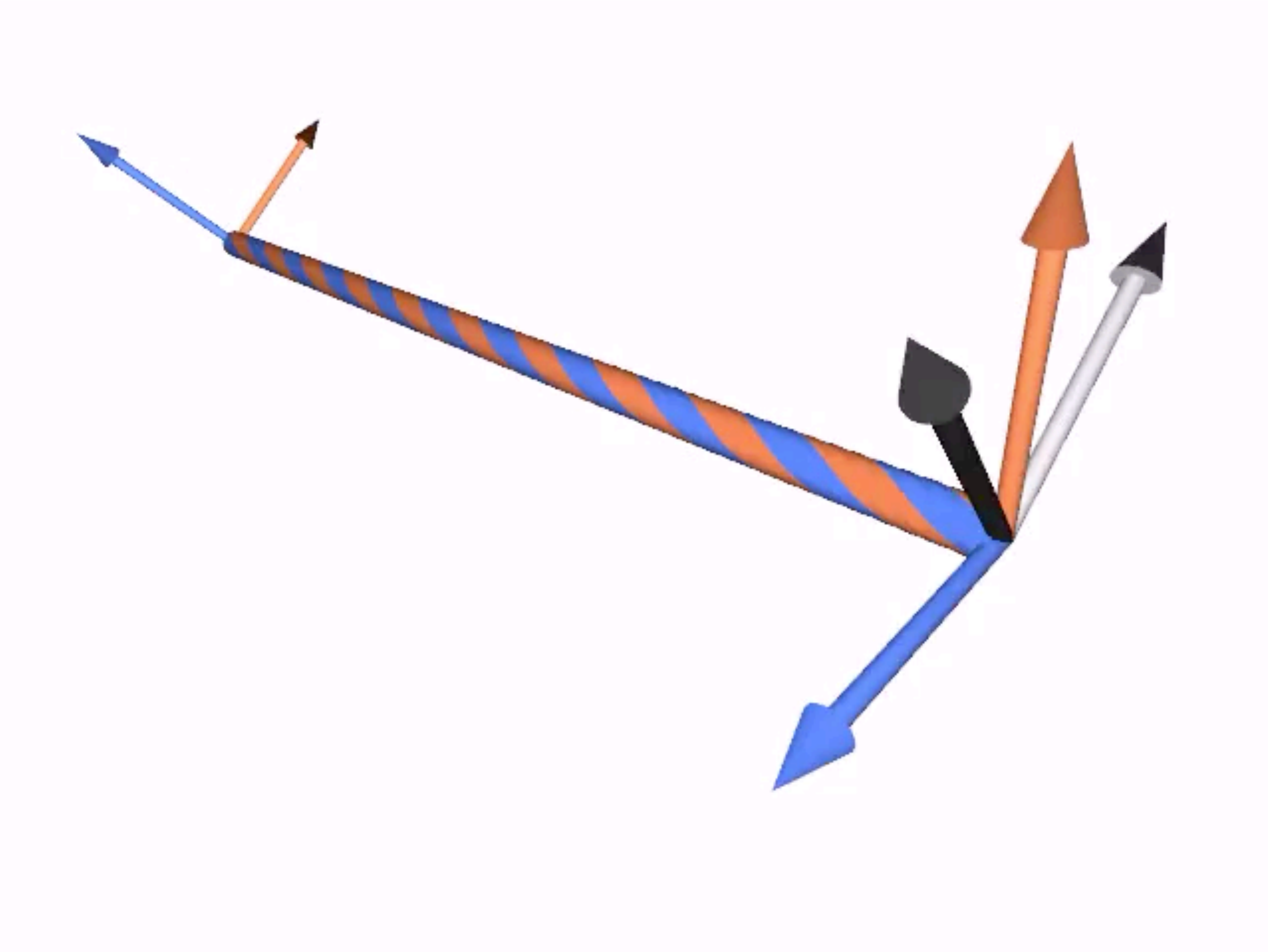
Bend & Twist Interaction



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Bend & Twist Interaction

$$\frac{1}{2}\int \alpha\boldsymbol{\kappa}^2 \mathrm{d}s + \frac{1}{2}\int \beta(\theta')^2 \mathrm{d}s$$

How do bending and twisting interact?

Both terms affect centerline

- bending force
  moves centerline toward straighter curve
- twisting force
  aligns twist-free & material frames

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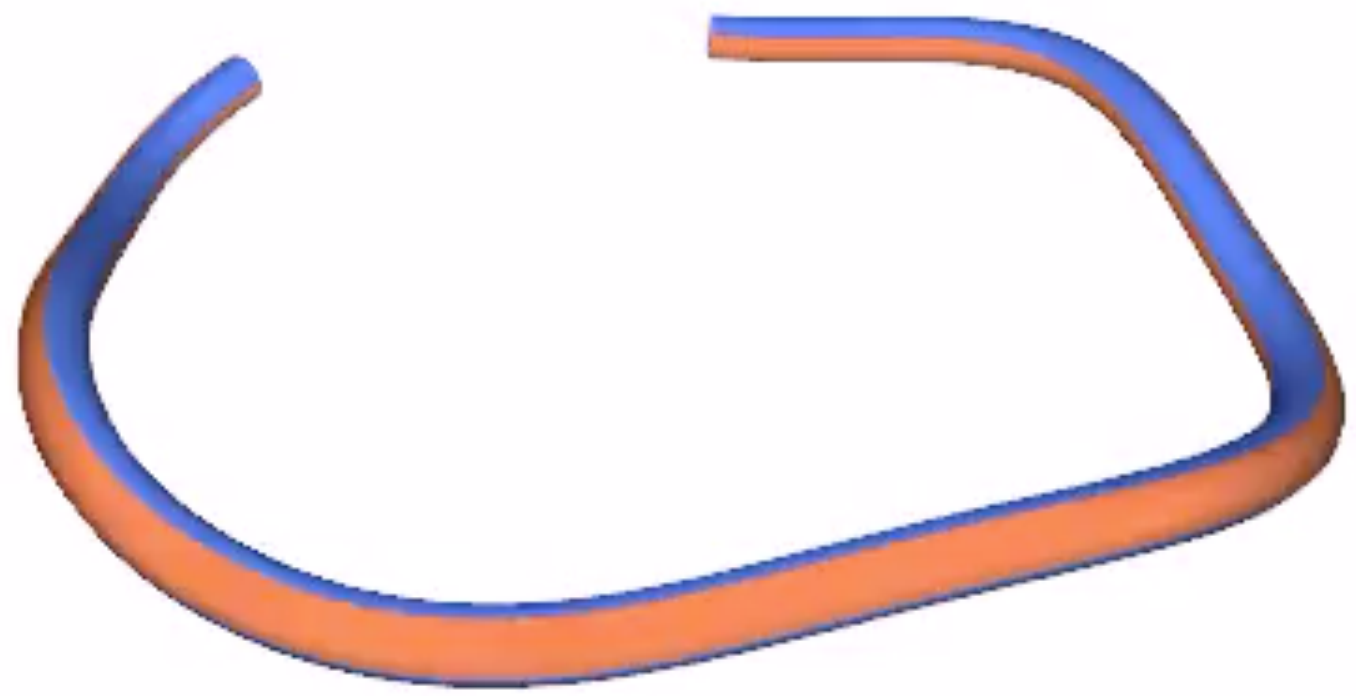
Quasistatic Material Frame

What is the speed of twist waves?

- depends on inertia of cross-section

Thin-rod limit



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Quasistatic Material Frame

What is the speed of twist waves?

- depends on inertia of cross-section

Thin-rod limit

- vanishing cross-section
- twist propagates instantly

Quasistatic material frame

- enforced during simulation

(slide by Miklós Bergou from *Discrete Elastic Rods* presentation)

Transition to Discrete Setting

Build model from discrete building blocks

- discrete curve + angle representation
- parallel transport
- twist
- curvature

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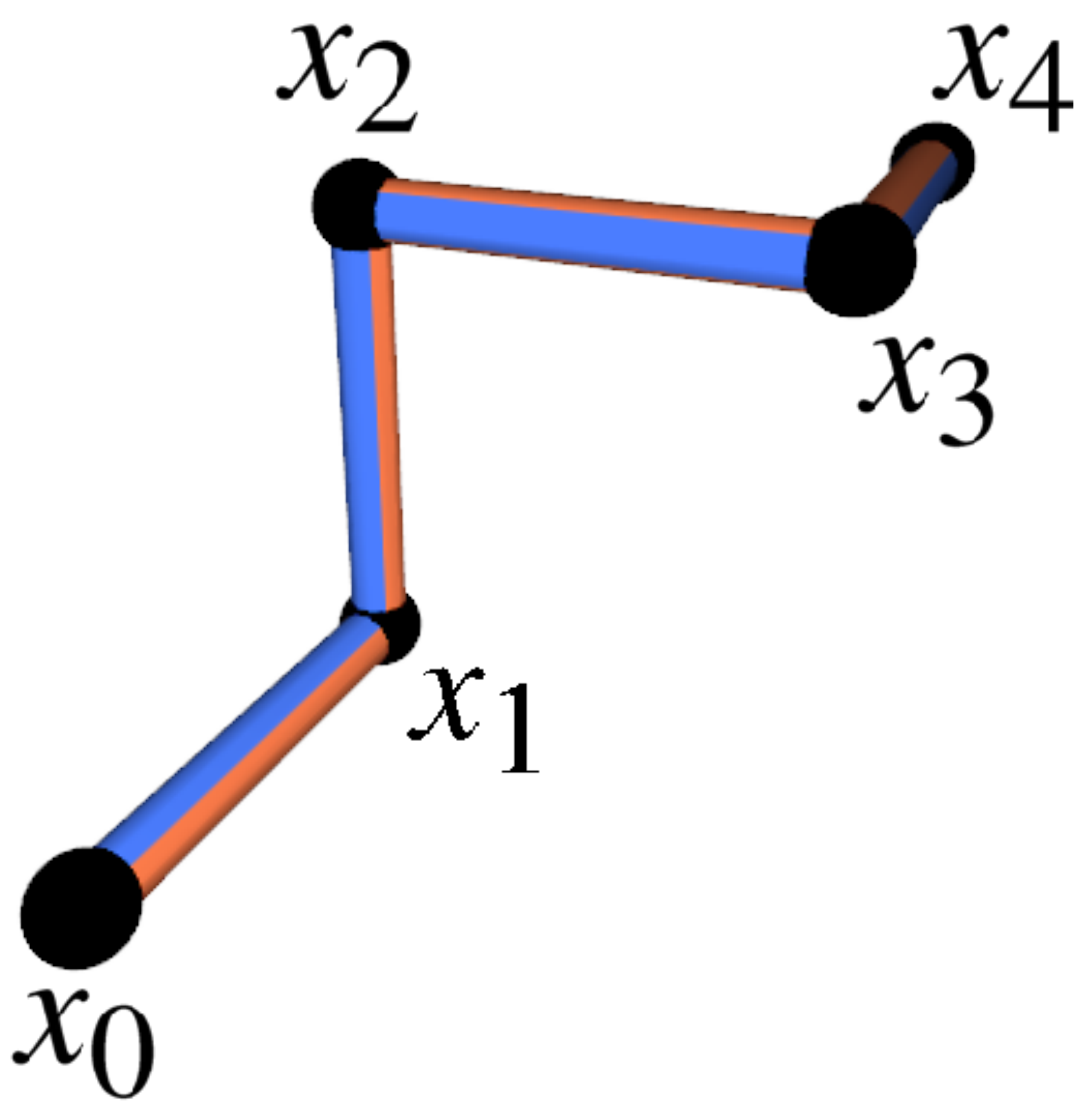
Discrete Curve + Angle

Explicit centerline



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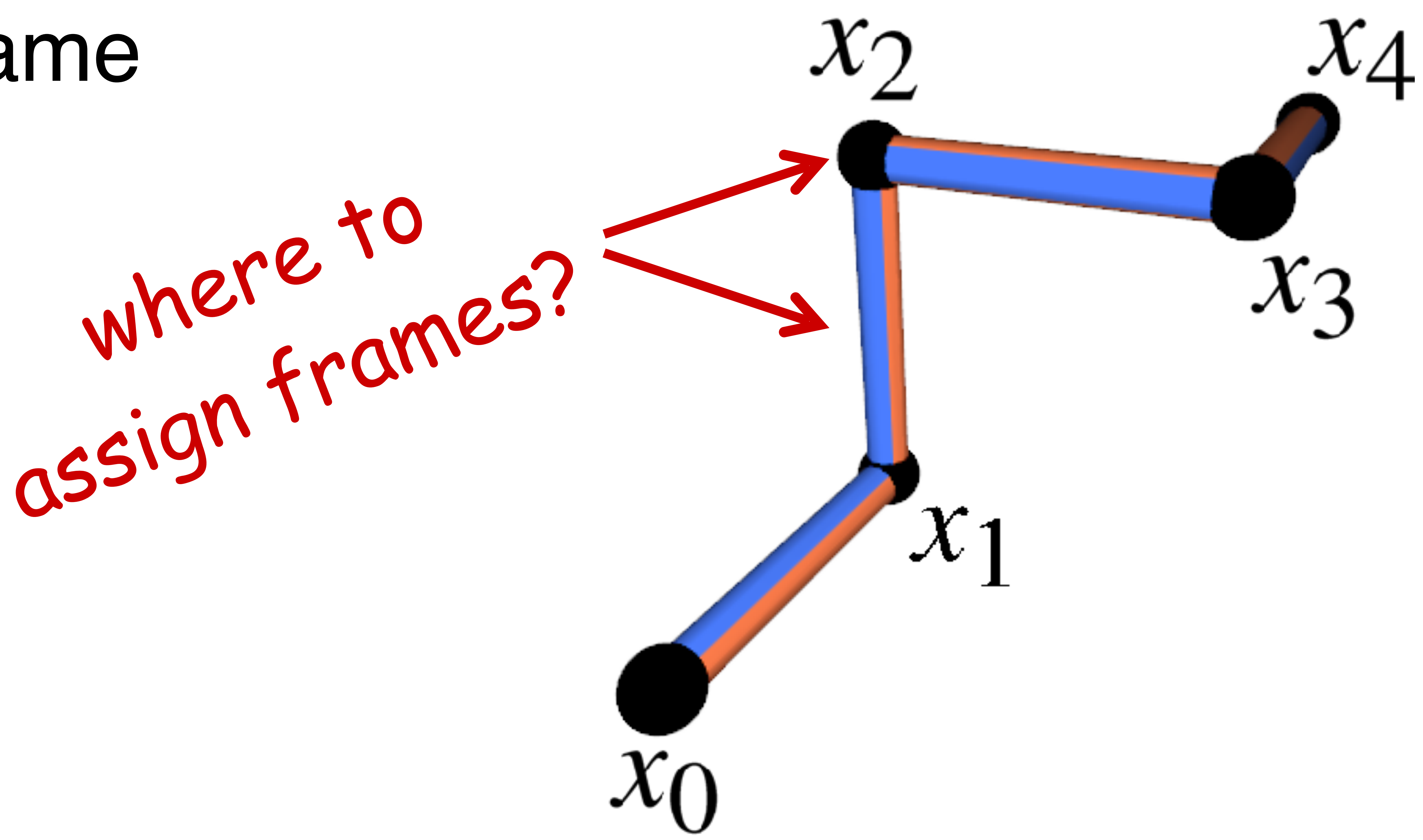
Discrete Curve + Angle

Explicit centerline

Reference frame



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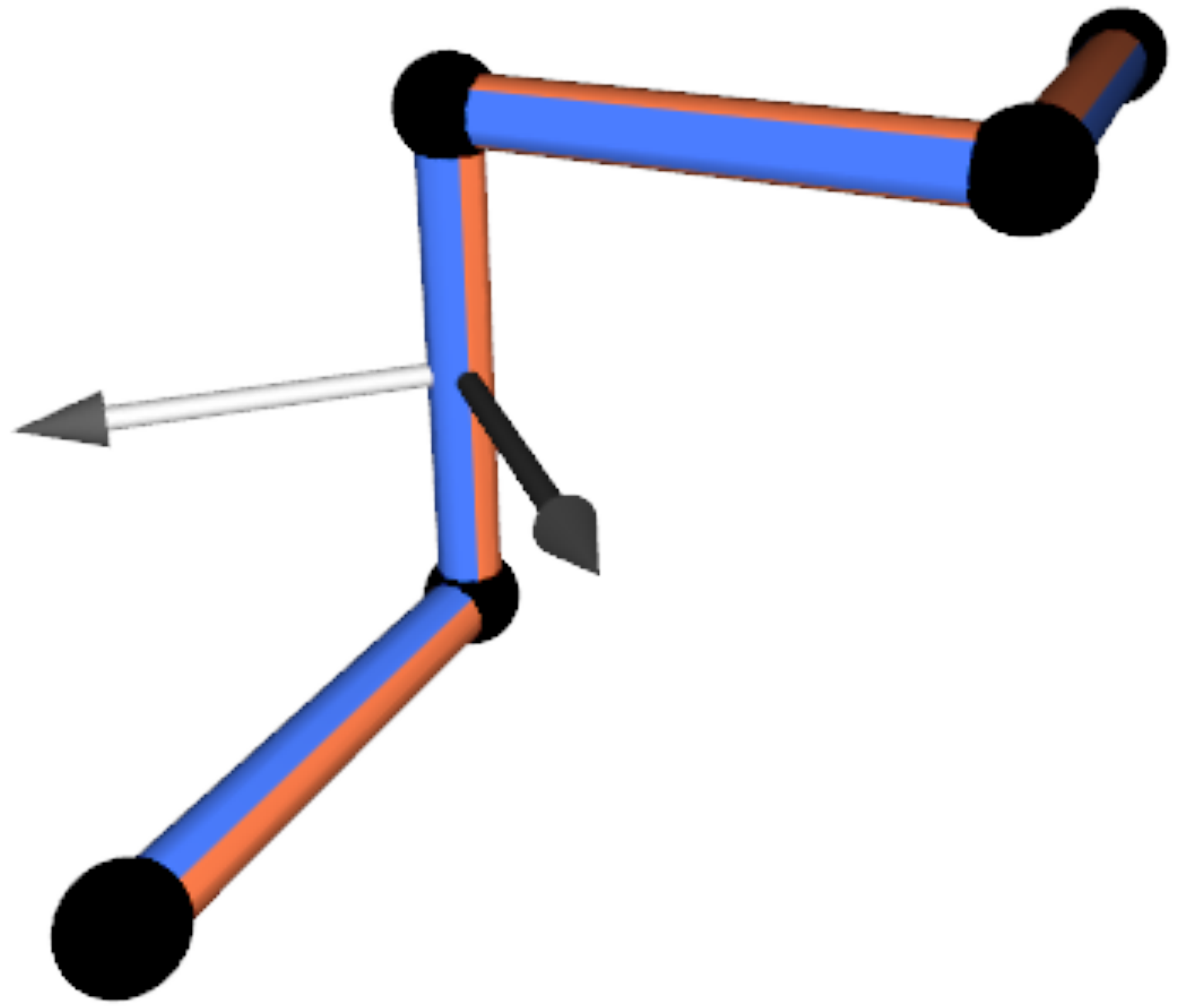
Discrete Curve + Angle

Explicit centerline

Reference frame

- $\{\mathbf{u}, \mathbf{v}\}$ twist-free



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Discrete Curve + Angle

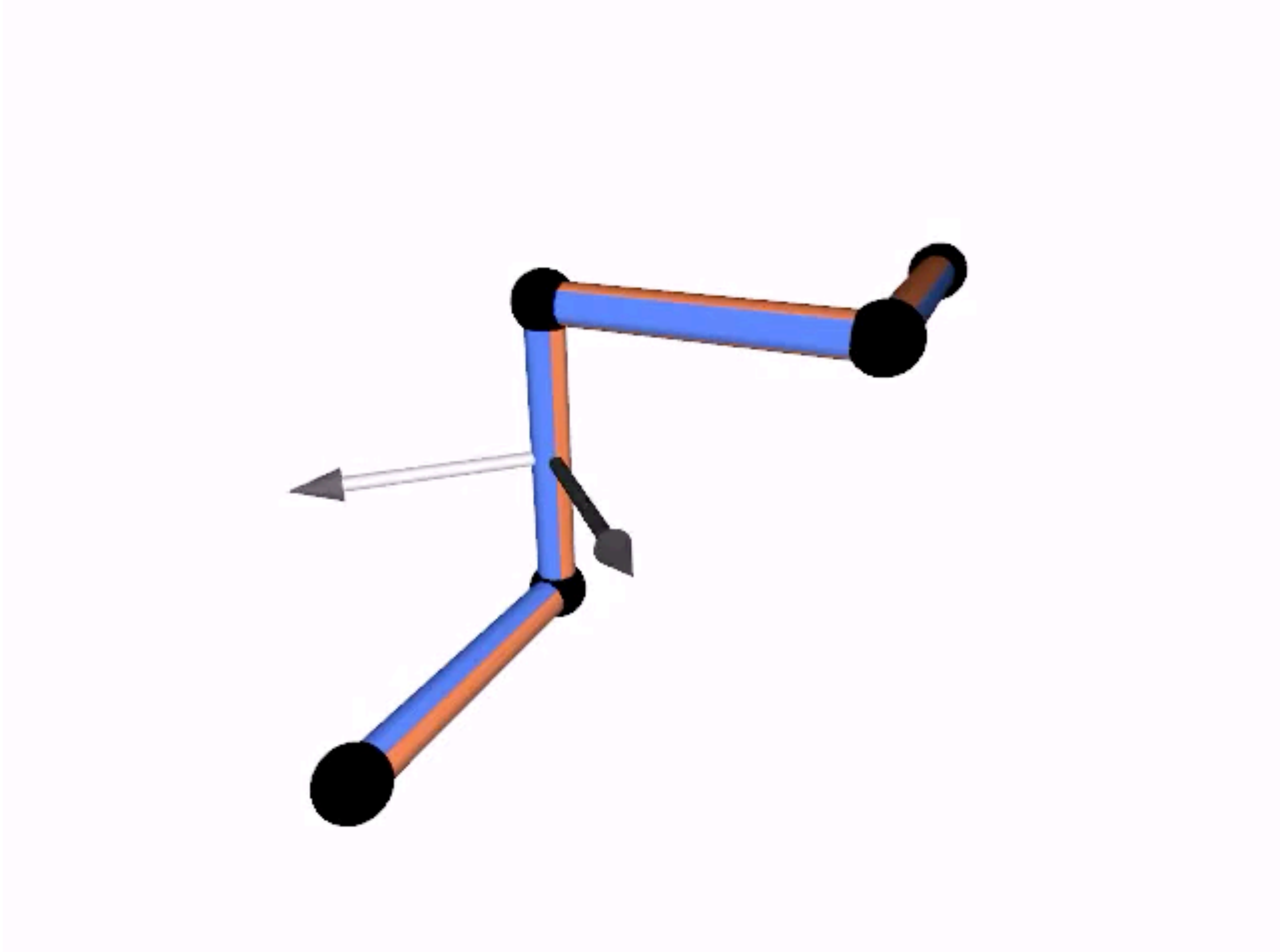
Explicit centerline

Reference frame

- $\{\mathbf{u}, \mathbf{v}\}$ twist-free

Material frame

- relative angle



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Discrete Curve + Angle

Explicit centerline

Reference frame

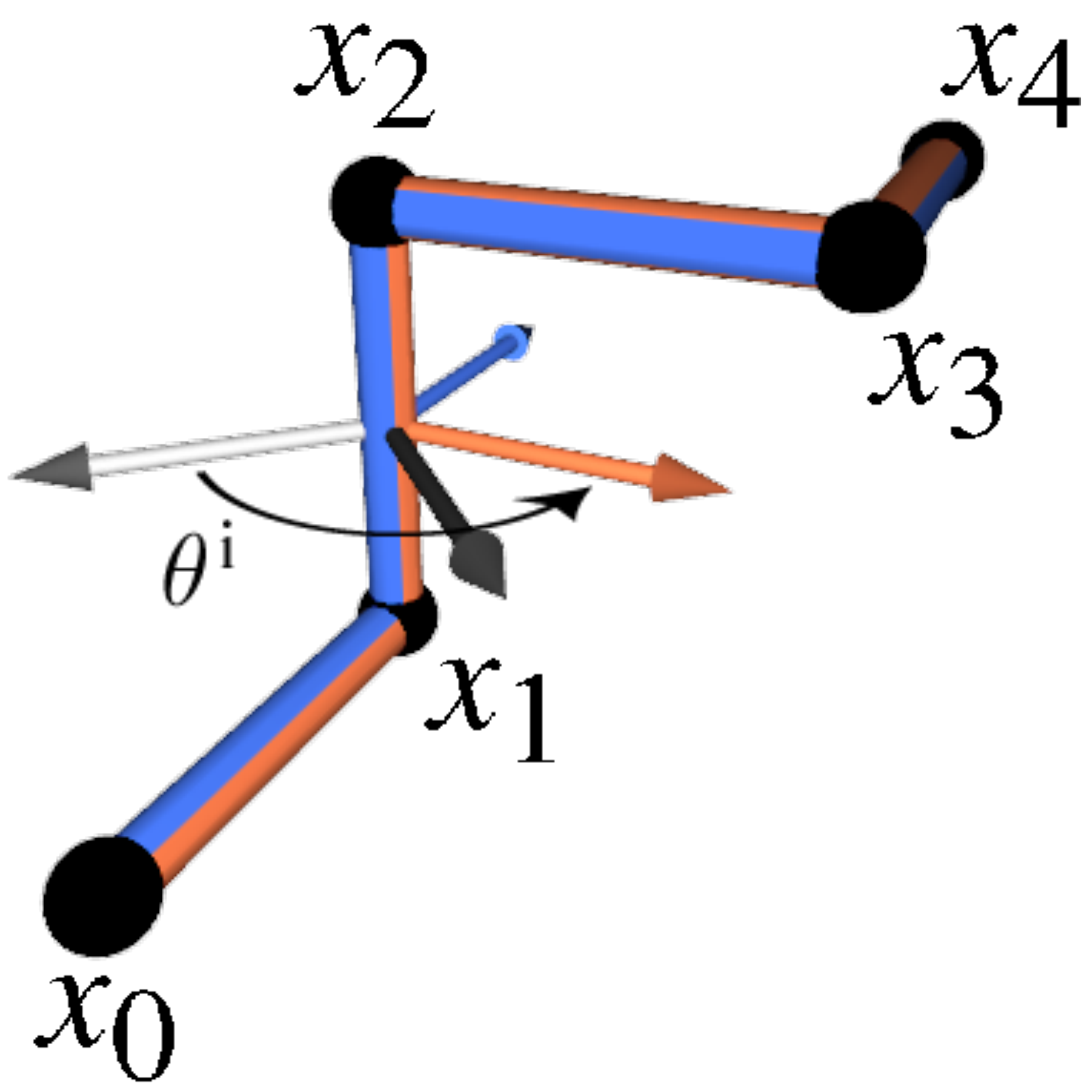
- $\{\mathbf{u}, \mathbf{v}\}$ twist-free

Material frame

- relative angle

Degrees of freedom

- $\{\mathbf{x}_0, \cdots, \mathbf{x}_{n+1}\} \in \mathbb{R}^{3(n+2)}$
- $\{\theta^0, \cdots, \theta^n\} \in \mathbb{R}^{n+1}$



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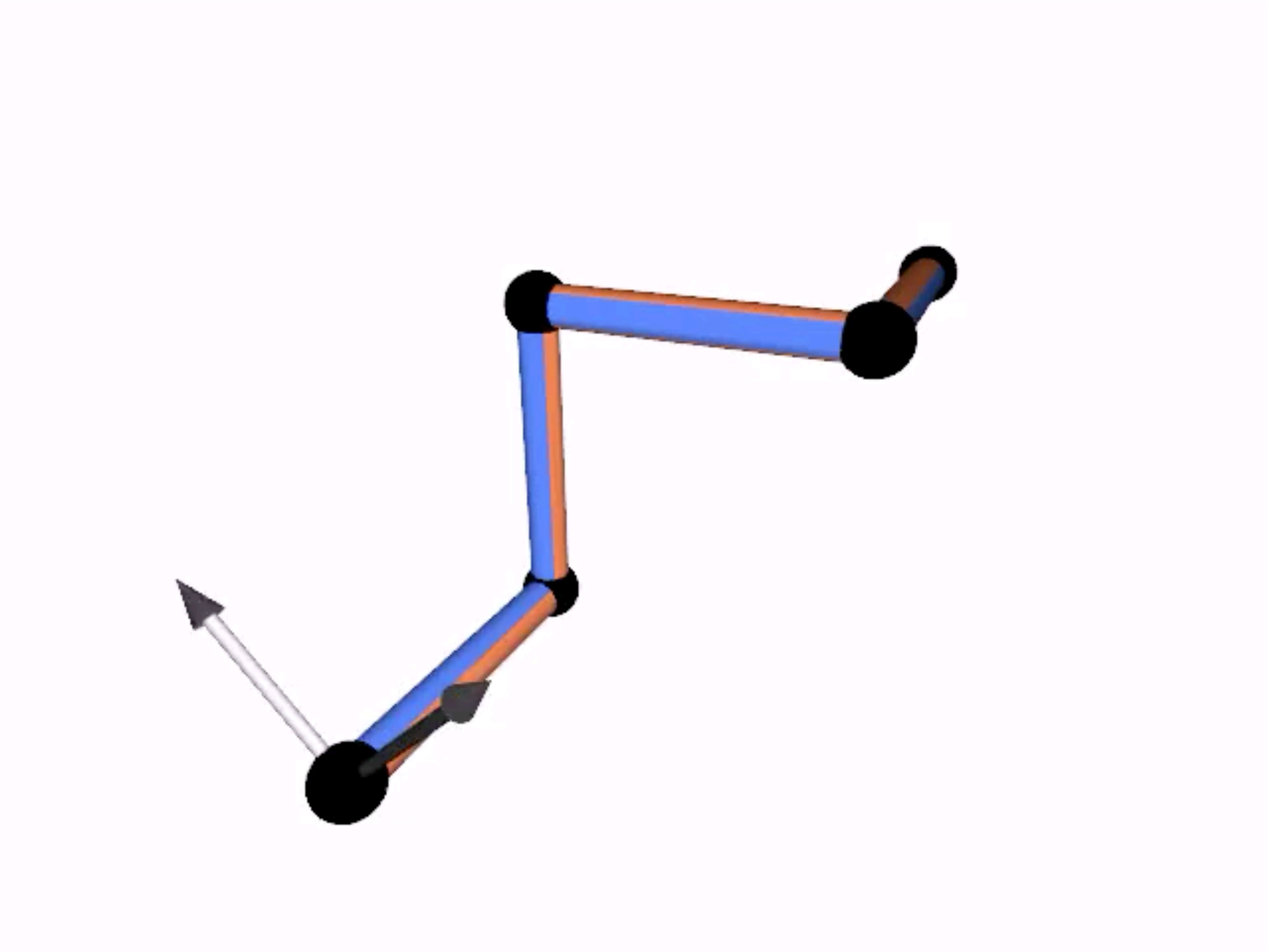
Twist-Free Frame



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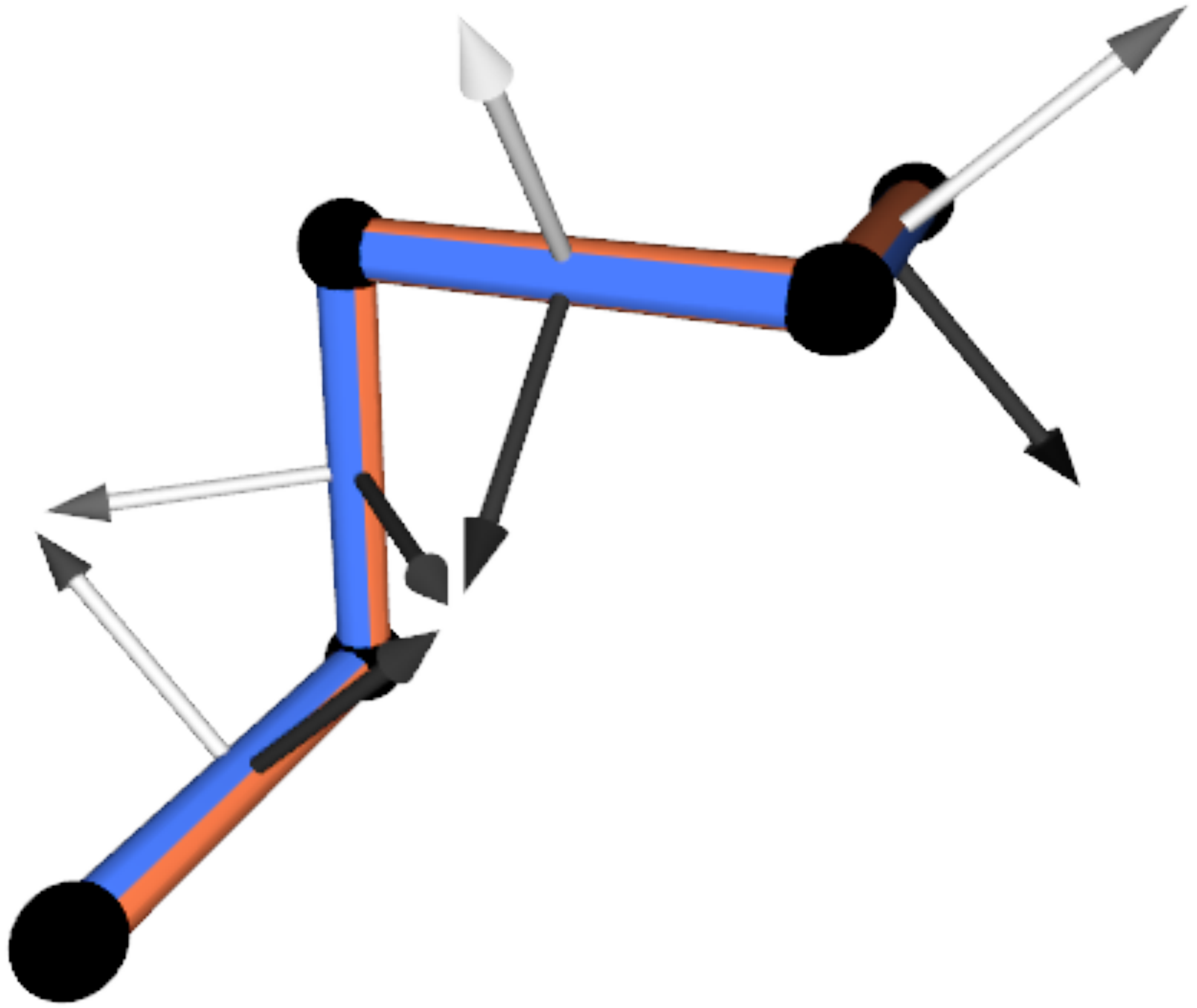
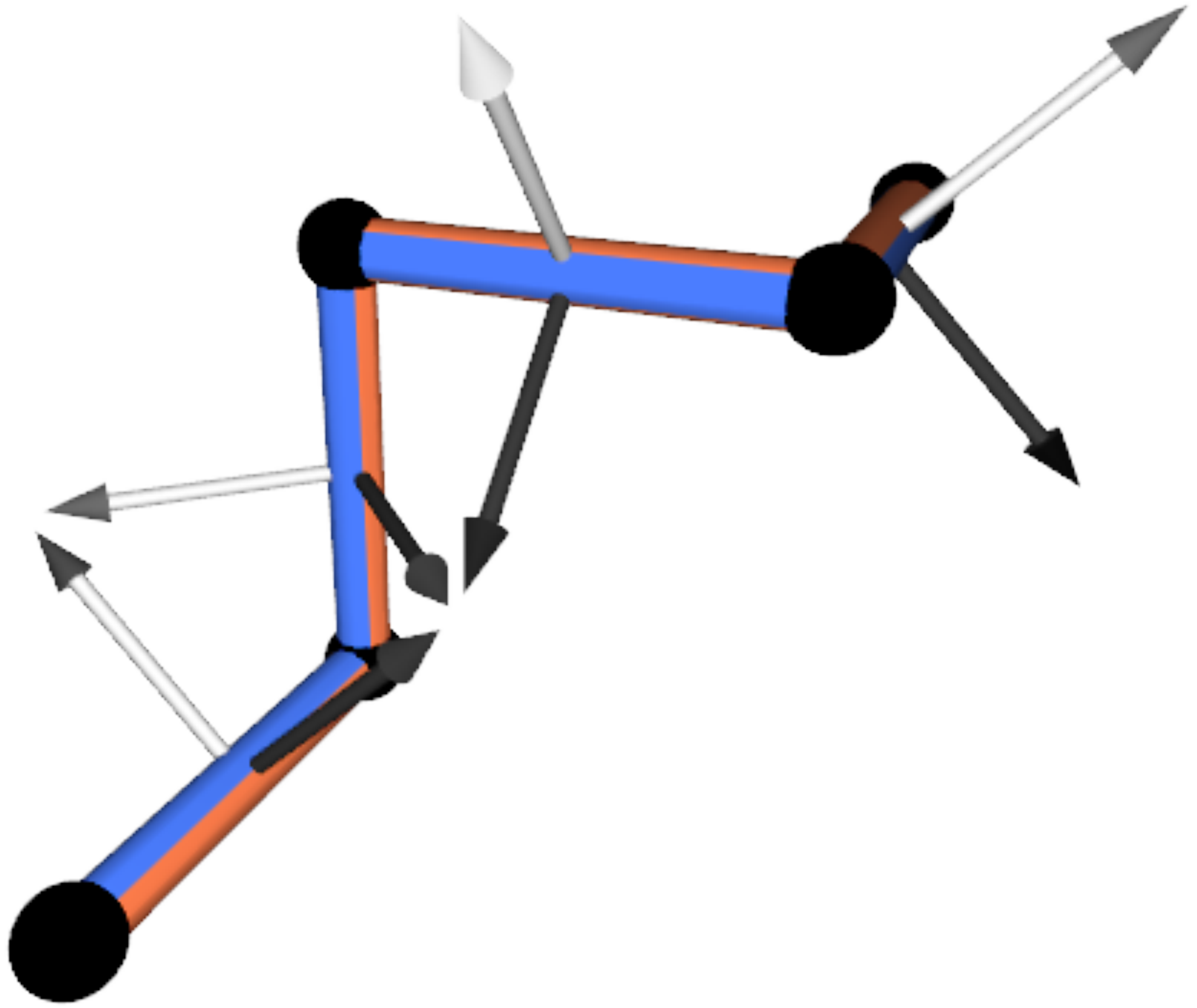


Requirements
1. no twist about tangent
2. frame must stay adapted

(slide by Miklós Bergou from *Discrete Elastic Rods* presentation)

Discrete Parallel Transport



Requirements
1. no twist about tangent
2. frame must stay adapted

Translate along edge
Rotate at vertex
about binormal

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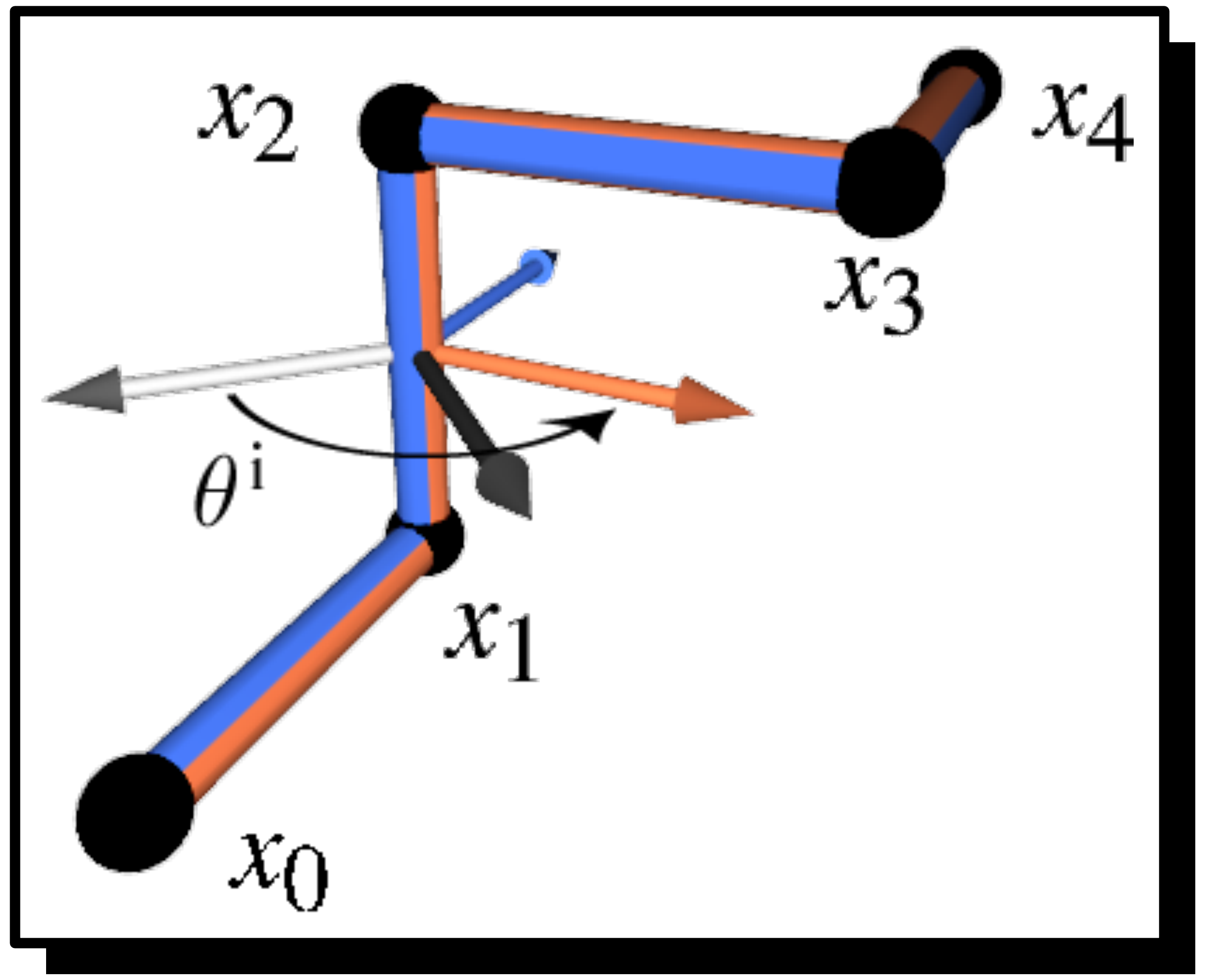
Discrete Energy



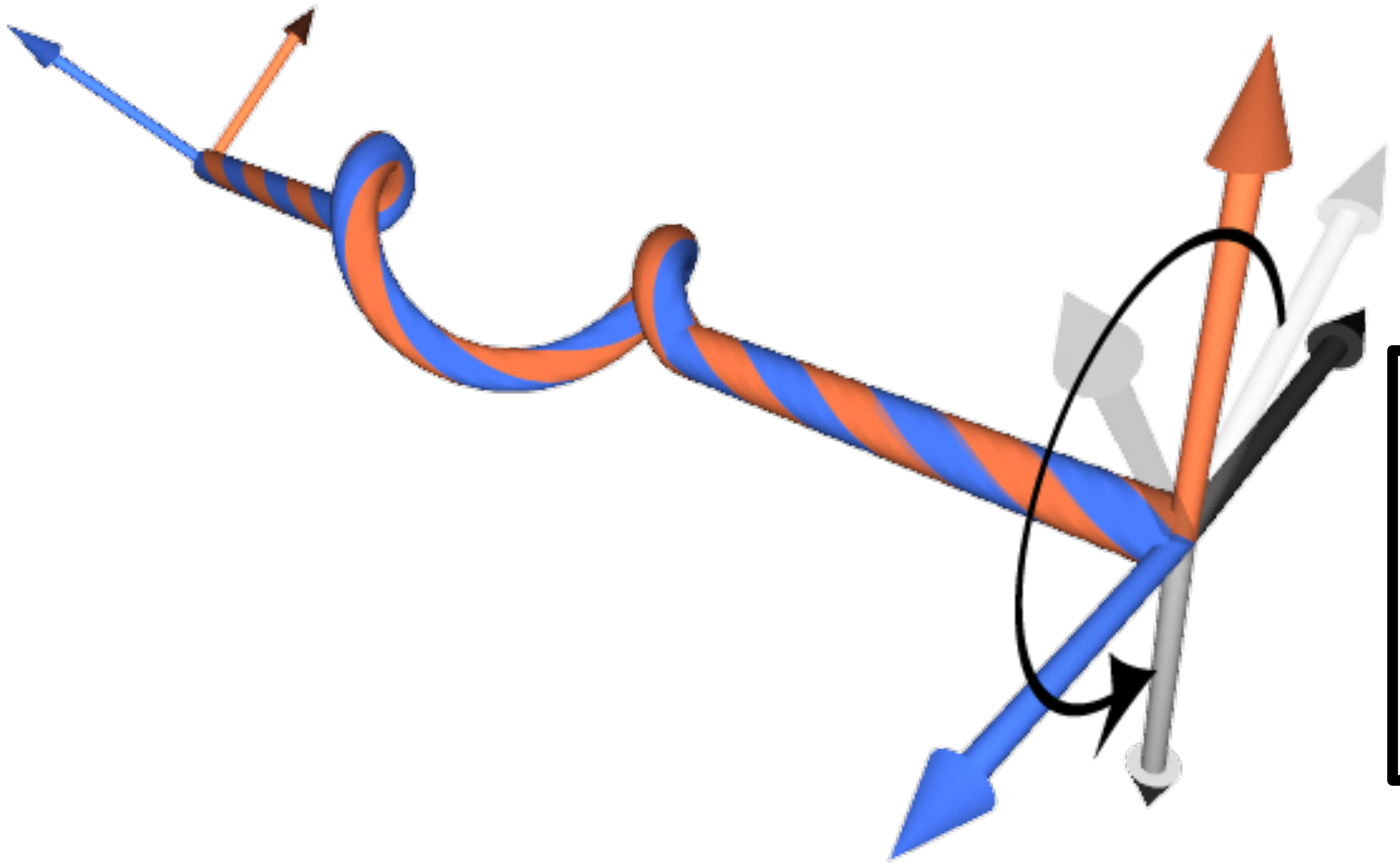
bending $\qquad\qquad$ twisting

$$\frac{1}{2}\int \alpha \boldsymbol{\kappa}^2 ds + \frac{1}{2}\int \beta (\theta')^2 ds$$

$$\boldsymbol{\kappa}_i = ? \qquad \theta_i' = \frac{\theta^i - \theta^{i-1}}{l_i}$$

(slide by Miklós Bergou from *Discrete Elastic Rods* presentation)

Curvature



change in angle

$$-\int \boxed{(\kappa \mathbf{b})} \cdot (\delta \mathbf{x})' \mathrm{d}s$$

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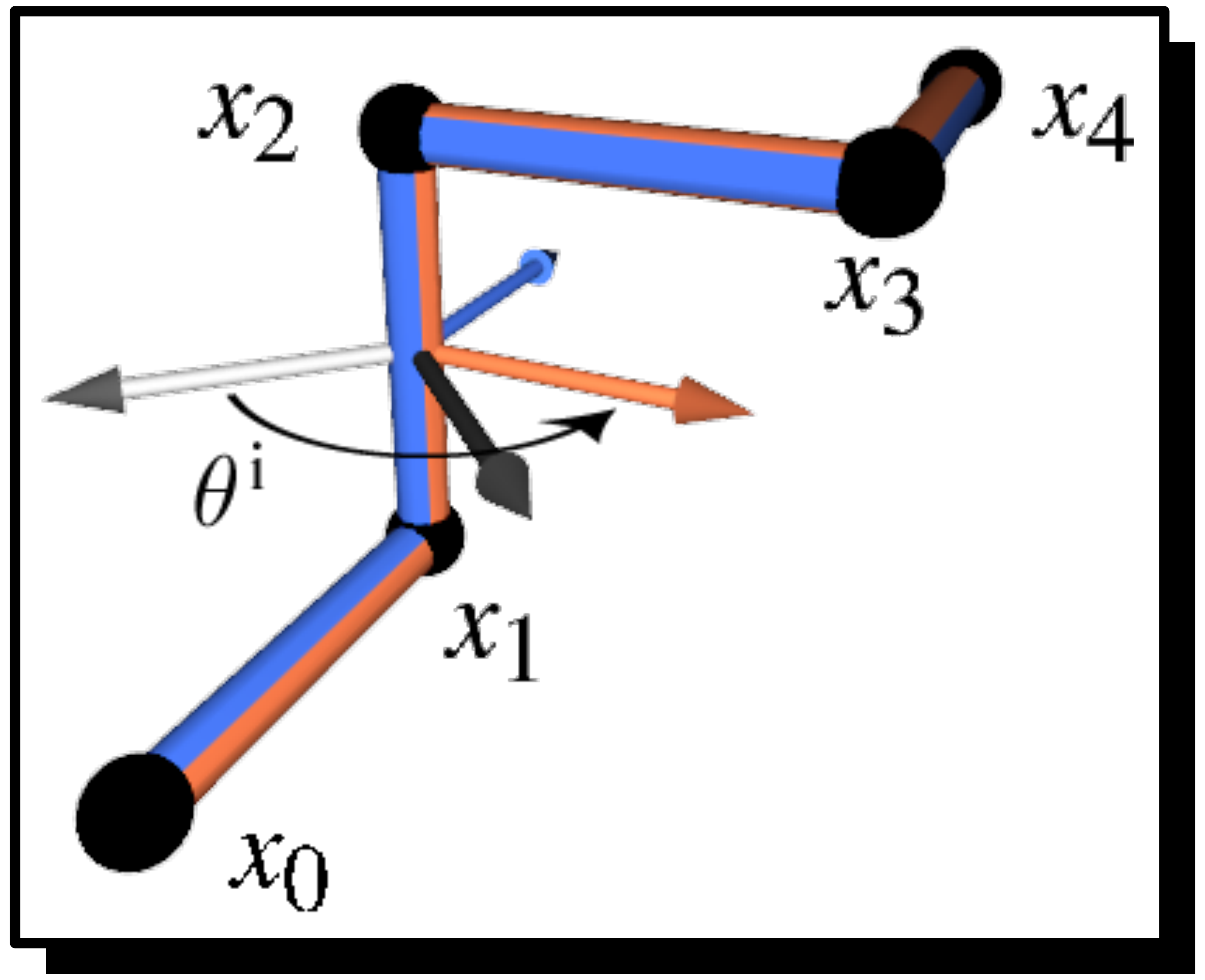
Discrete Energy



bending   twisting

$$\frac{1}{2}\int \alpha \boldsymbol{\kappa}^2 ds + \frac{1}{2}\int \beta (\theta')^2 ds$$

$$\kappa_i = 2\tan\left(\frac{\phi_i}{2}\right) \qquad \theta_i' = \frac{\theta^i - \theta^{i-1}}{l_i}$$

(slide by Miklós Bergou from *Discrete Elastic Rods* presentation)

Quasistatic Postulation

Quasistatic material frame update

$$\frac{\mathrm{d}E}{\mathrm{d}\theta^j} = 0 \qquad n \text{ equations in } n \text{ unknowns}$$

Computation greatly simplified

- larger time steps
- less computation

Twist uniform for isotropic rods

(slide by Miklós Bergou from *Discrete Elastic Rods* presentation)

(slide by Miklós Bergou from *Discrete Elastic Rods* presentation)

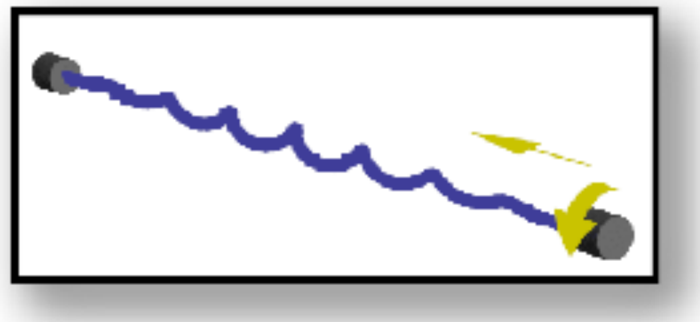
Putting It All Together

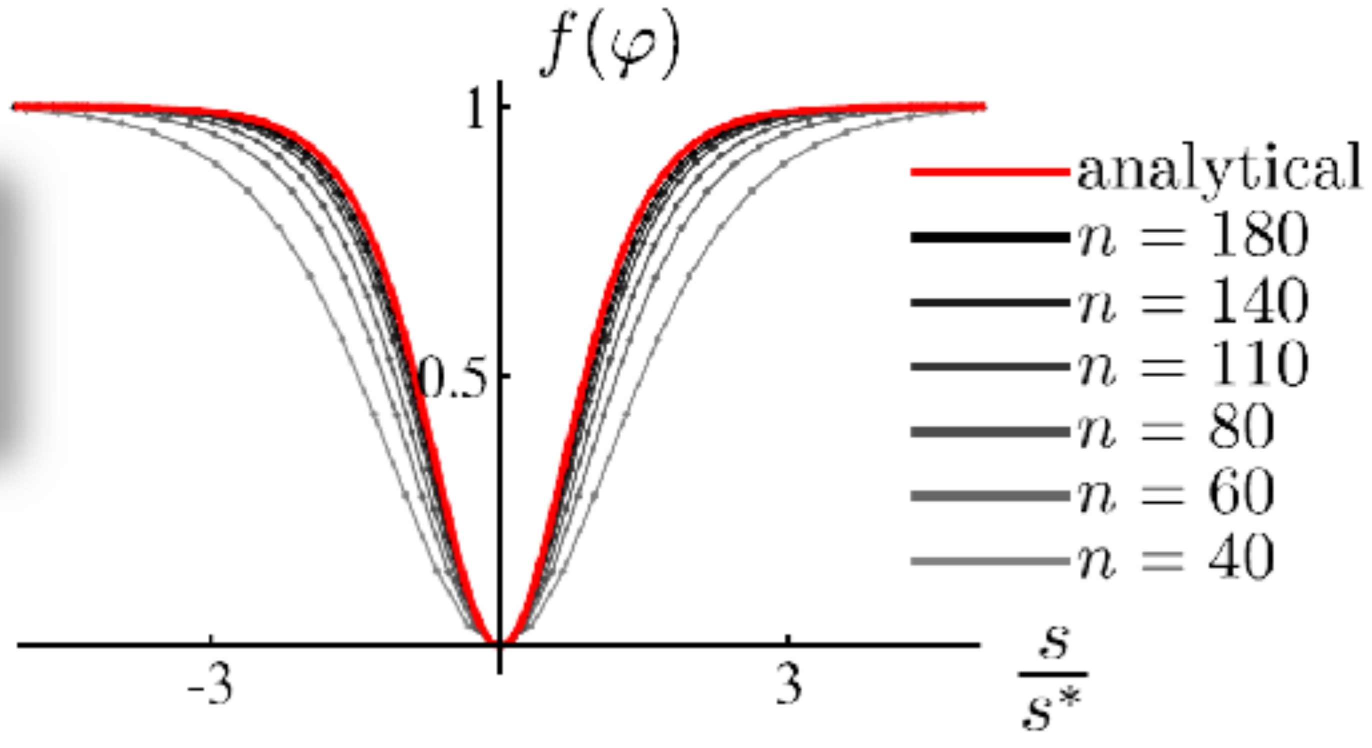
1. Update twist-free frame
   - parallel transport
2. Update material frame
   - quasistatic solve $\frac{\mathrm{d}E}{\mathrm{d}\theta^j} = 0$
3. Compute centerline forces
   - gradient of energy (+ gravity, damping, …)
4. Update centerline positions

(slide by Miklós Bergou from *Discrete Elastic Rods* presentation)

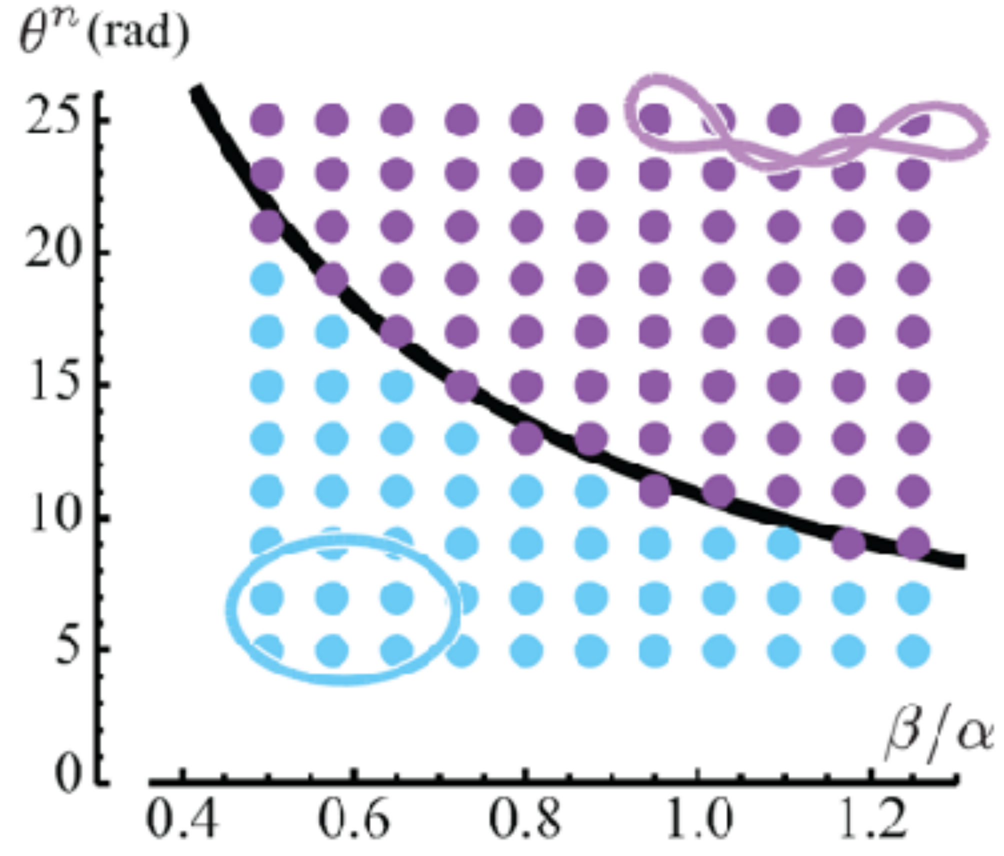
Convergence

Modulated helix





Michell's instability



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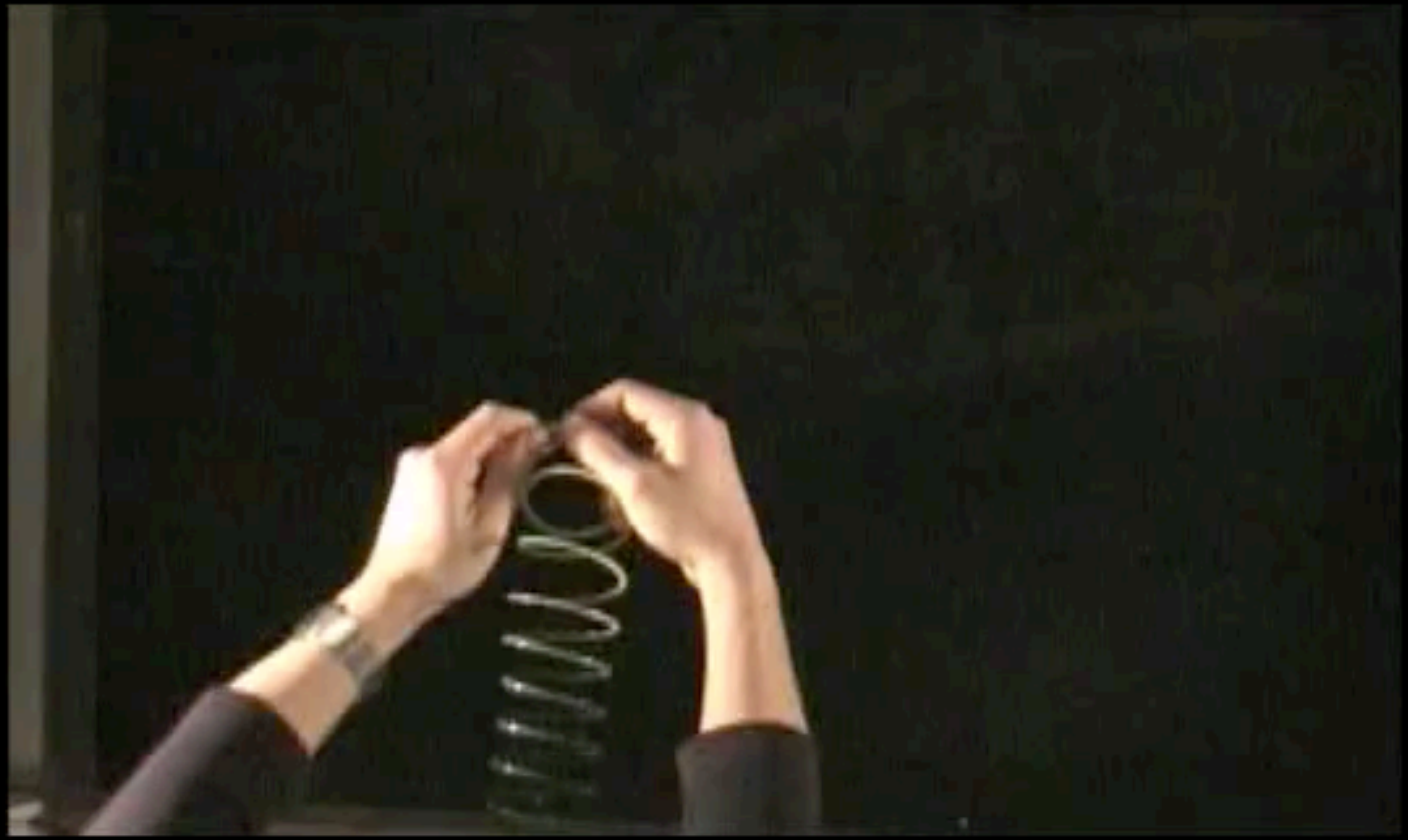

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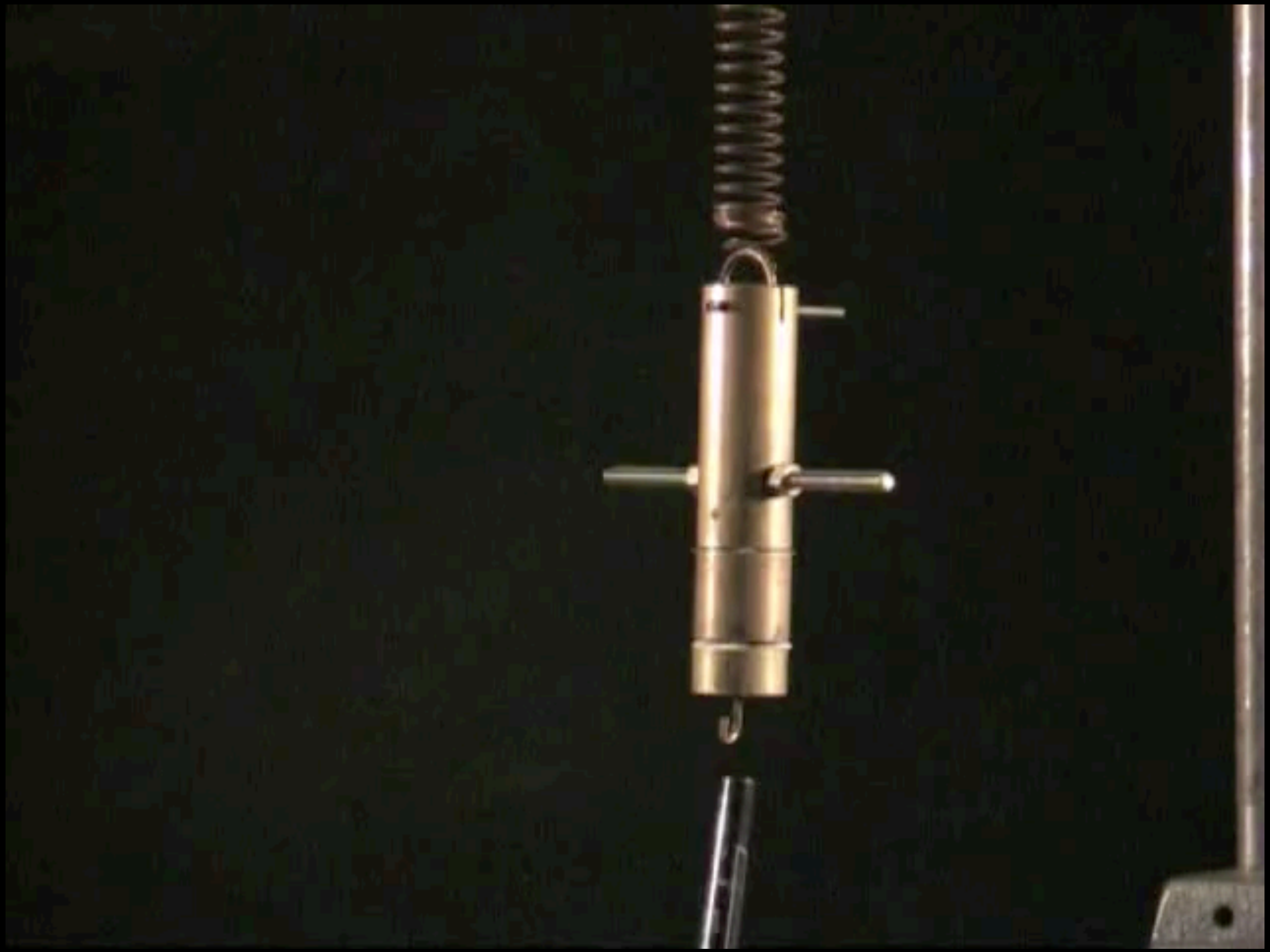

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