CS**5643**

13 Position-based Dynamics

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Constrained dynamics

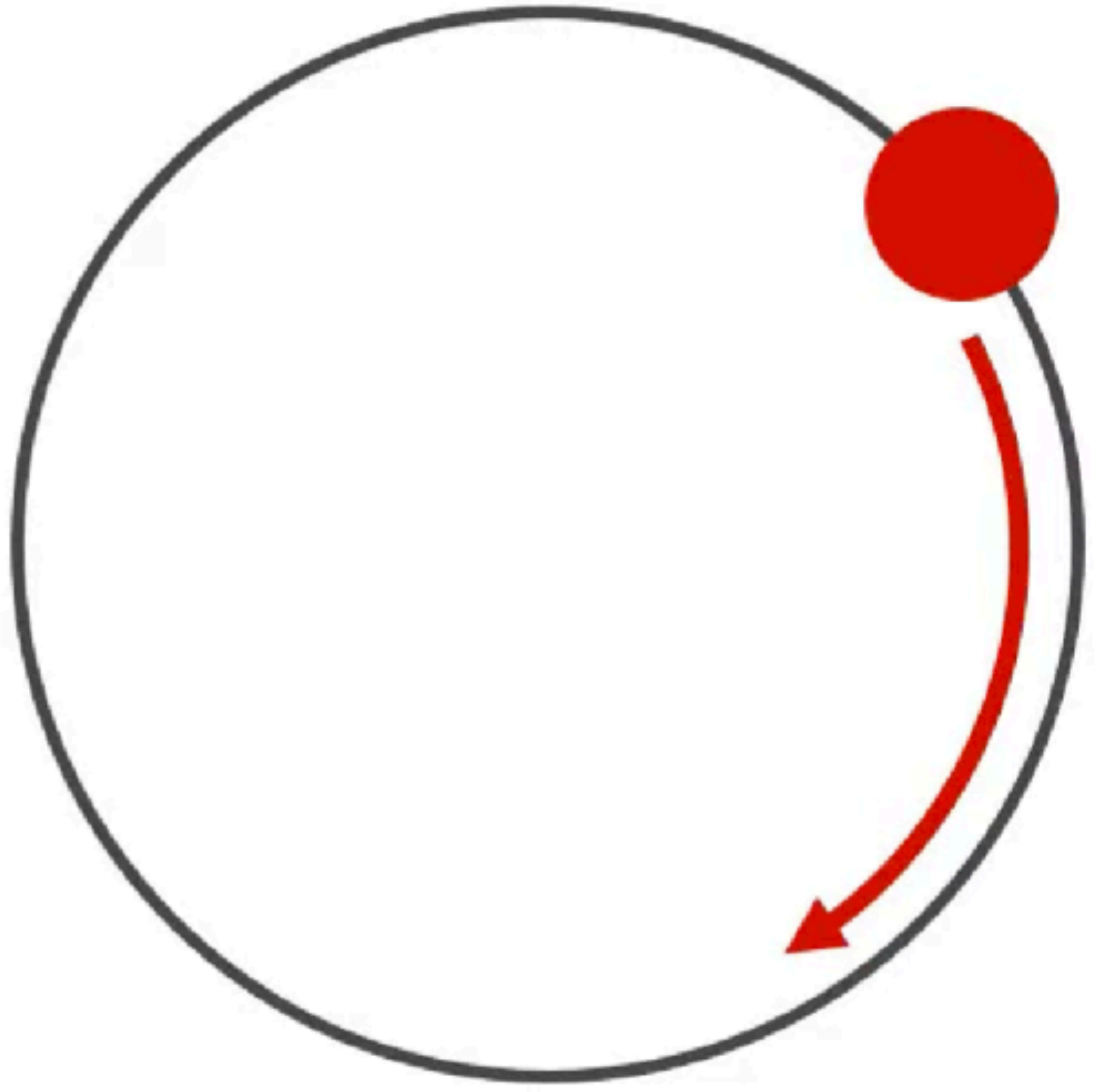
Objects affected by forces as always, but also constraints

$$M\ddot{\mathbf{x}} = f(\mathbf{x}, \dot{\mathbf{x}}) + \underbrace{\nabla C(\mathbf{x})^T \lambda}$$

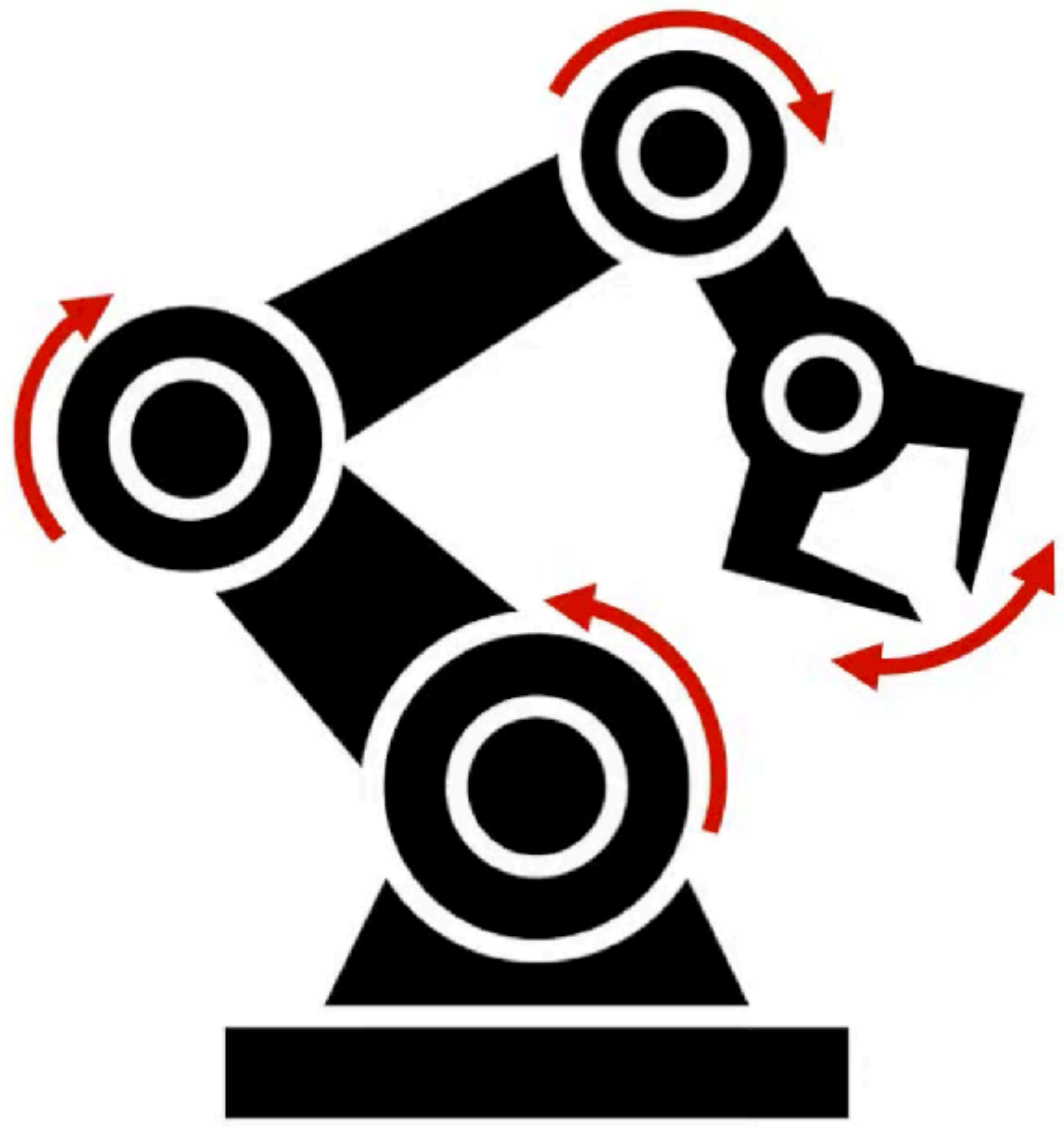
$$C(\mathbf{x}) \equiv \mathbf{0}$$

passive constraint forces (do no work)

Motion of objects is restricted



1 degree of freedom (dof)



4 dofs

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Methods for solving constraints

Springs

- pull system state towards $C = \mathbf{0}$
- must be stiff to be effective, leads to numerical stiffness

Reduced coordinates

- parameterize the set of feasible states—great for special problems, hard to use in general

Constraint forces

- solve for forces that ensure $C = \dot{C} = \ddot{C} = \mathbf{0}$
- works great but requires solving global systems

Post-step projection

- first compute timestep ignoring constraints, then fix the constraints

Position-based dynamics

A post-step projection type method

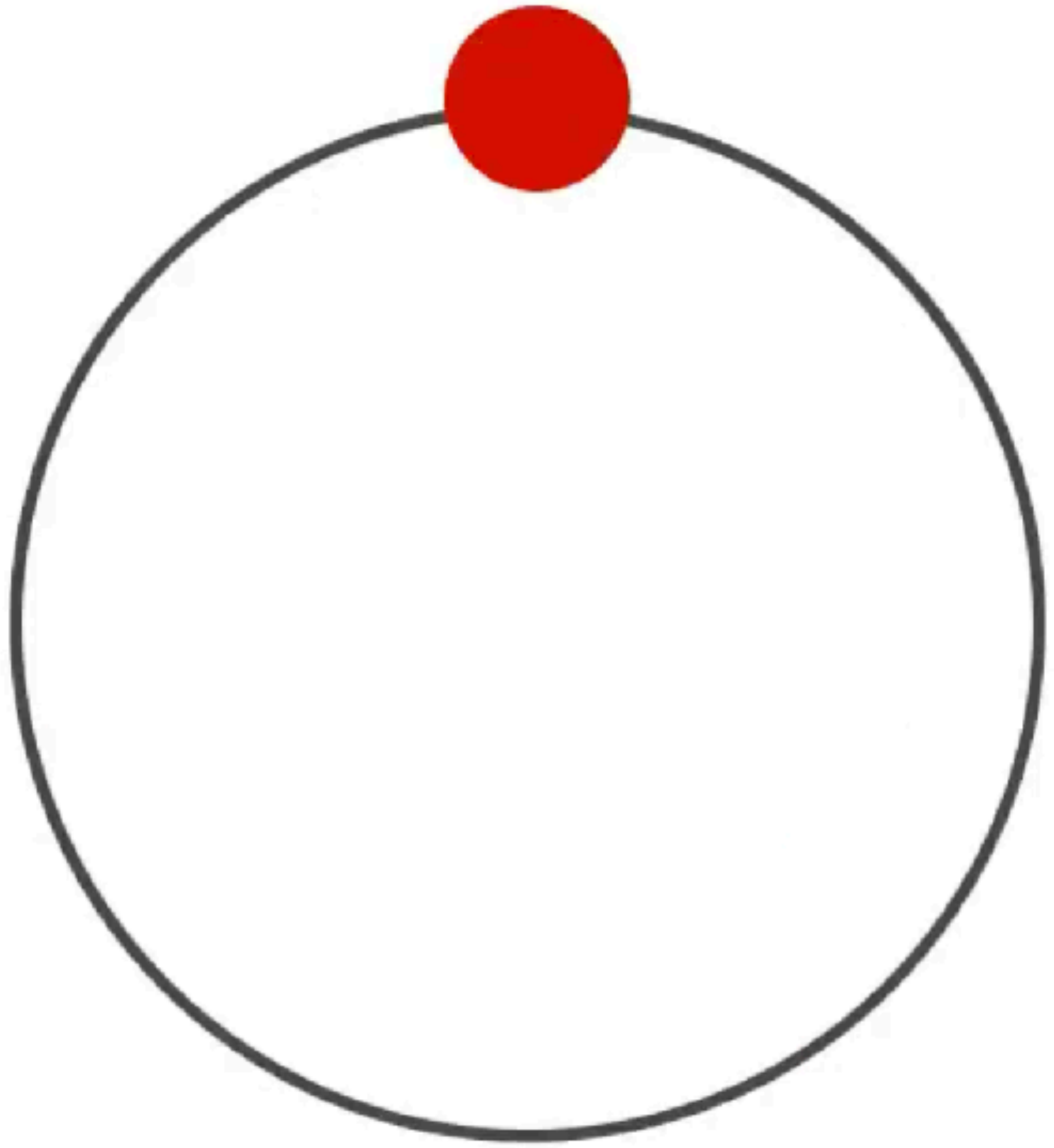
Carefully chooses methods for simple implementation

- projects end-step positions onto constraints then defines velocity to fit the projected positions
- solves for positions using Gauss-Seidel iteration
- computing a G-S iteration turns out to involve simple per-constraint steps
- degrades gracefully for infeasible constraints
- can simulate deformable objects by just not letting the iteration converge

PBD history

- Müller et al. 2006 “PBD” introduced basic mechanism
- Macklin et al. 2016 “XPBD” cleaned it up and improved handling of deformables
- Macklin et el. 2019 “Small Steps” further refined the solver iteration

The simplest way to fix the velocity



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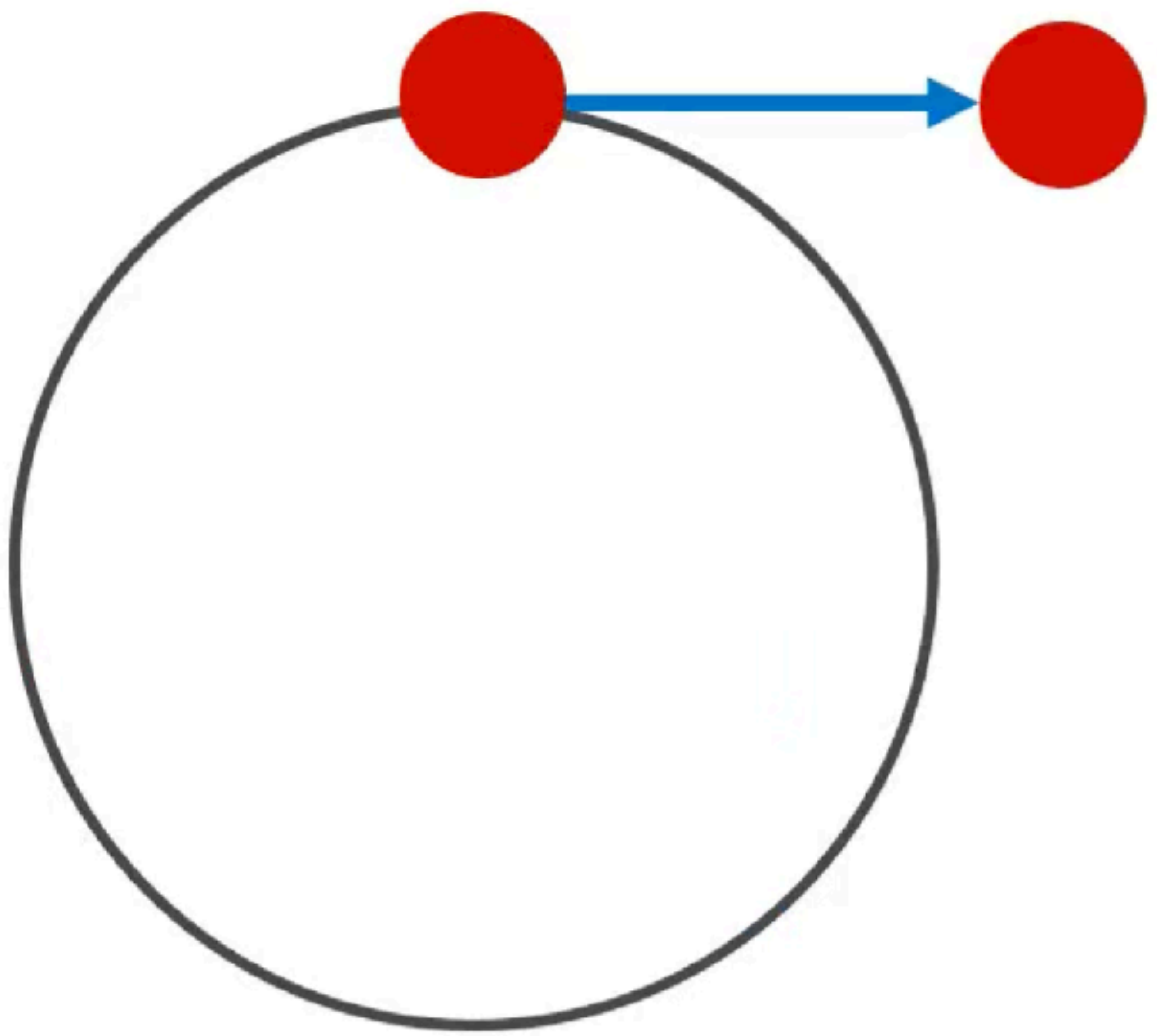
The simplest way to fix the velocity



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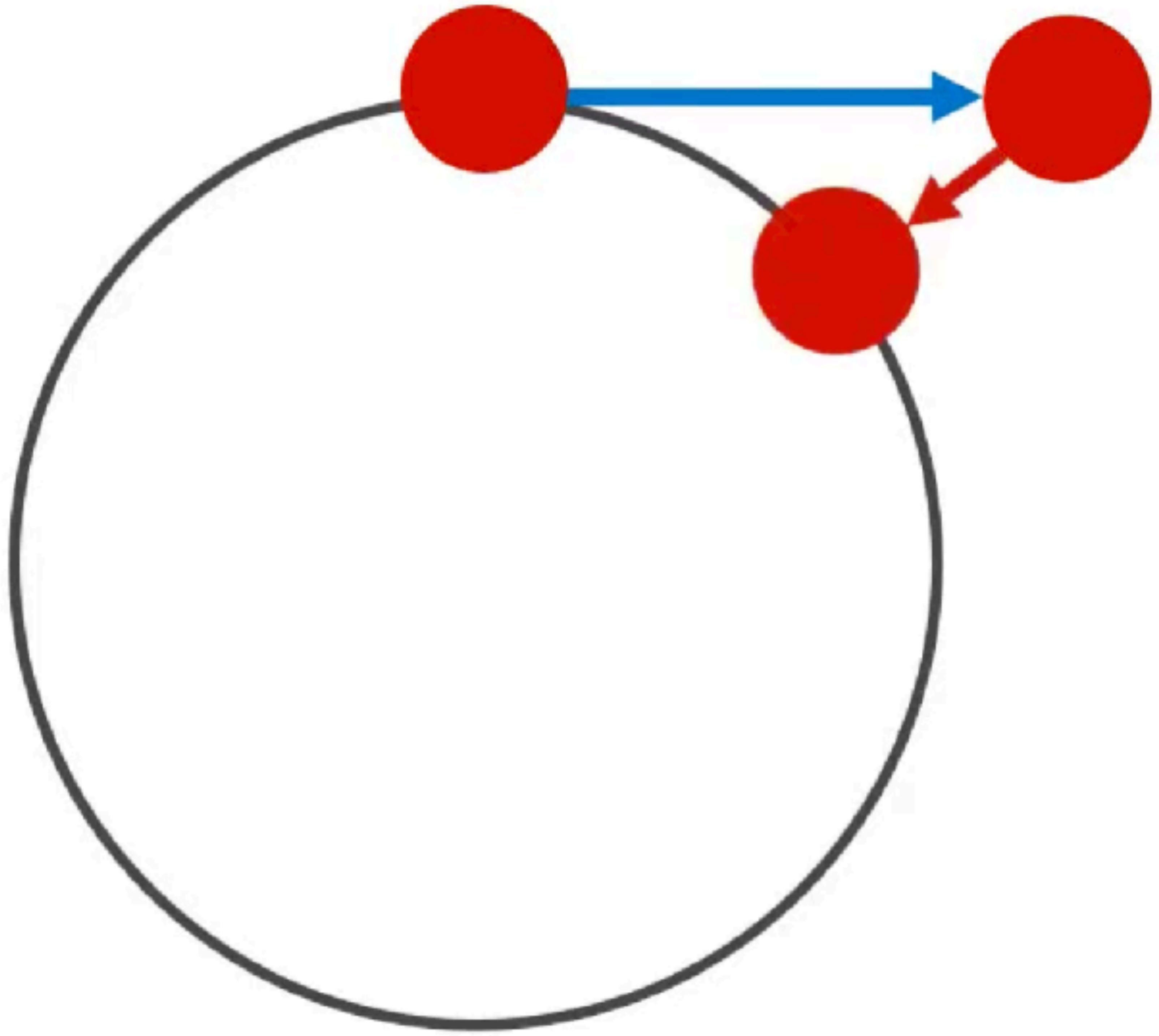
The simplest way to fix the velocity



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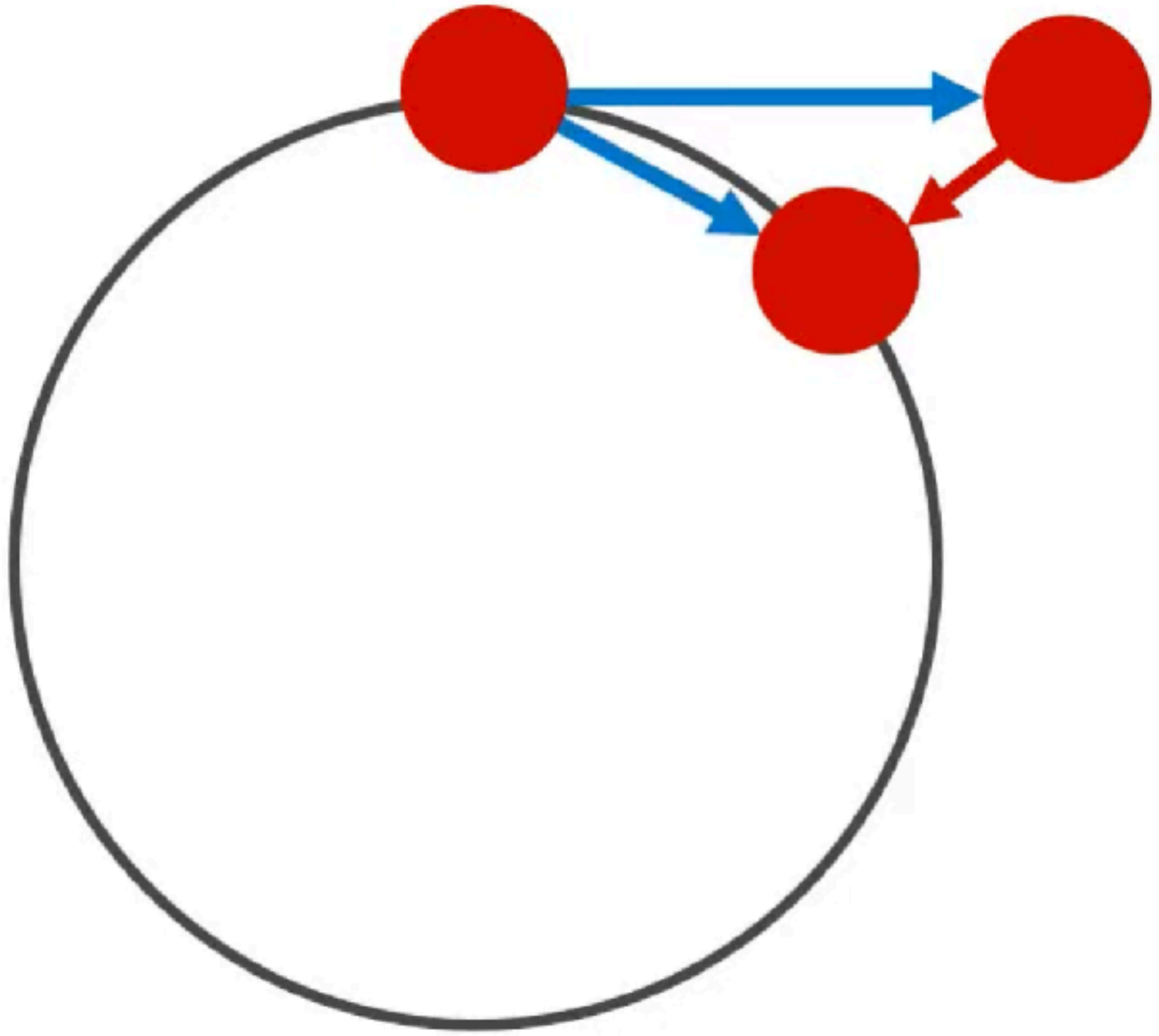
The simplest way to fix the velocity



velocity = (position at the end of the time step
- position at the beginning of the time step) / dt

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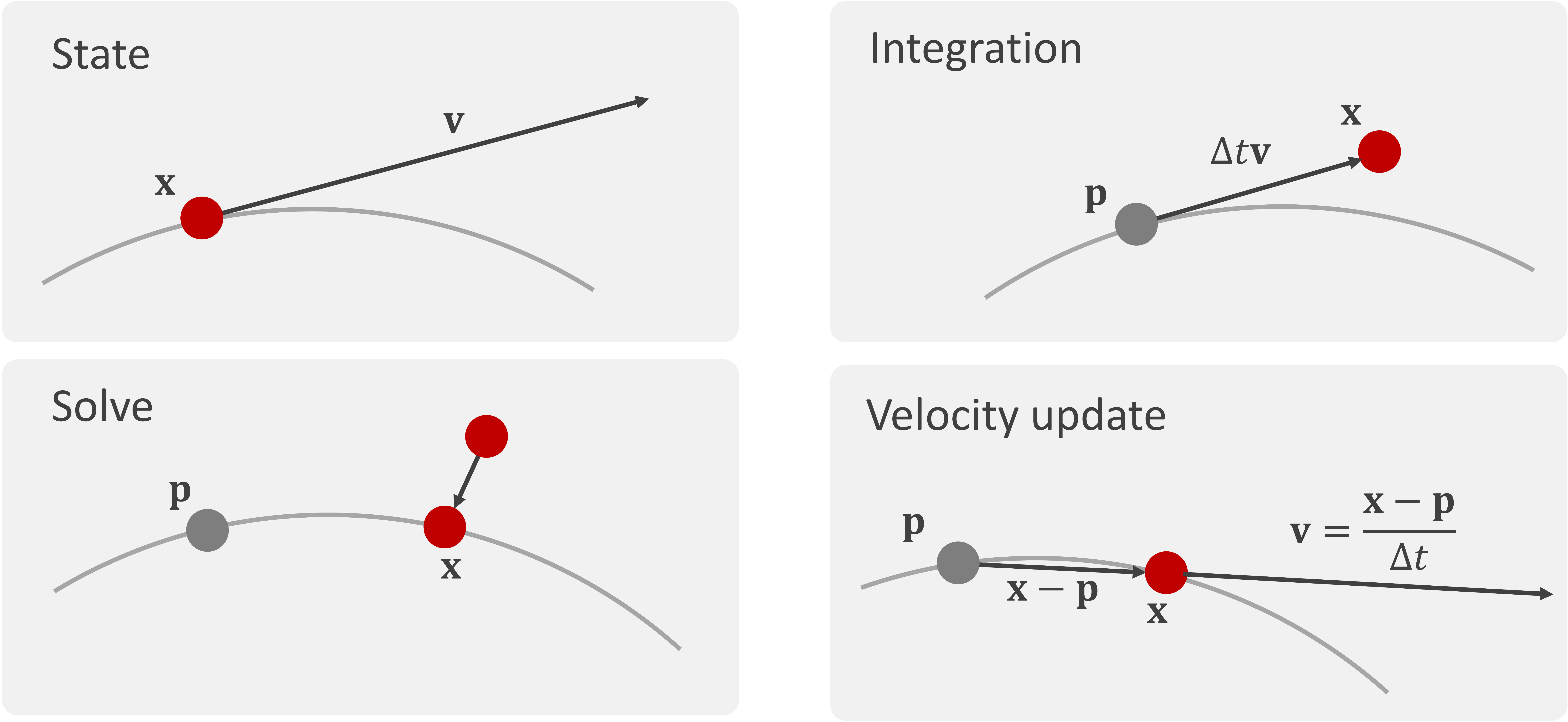
Bead on a Wire



PBD = integrator and solver!

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PBD Algorithm

while simulating
  for all particles i
    $\mathbf{v}_i \leftarrow \mathbf{v}_i + \Delta t\ \mathbf{g}$
    $\mathbf{p}_i \leftarrow \mathbf{x}_i$
    $\mathbf{x}_i \leftarrow \mathbf{x}_i + \Delta t\ \mathbf{v}_i$

  for all constraints C
    solve($C, \Delta t$)

  for all particles i
    $\mathbf{v}_i \leftarrow (\mathbf{x}_i - \mathbf{p}_i)/\Delta t$

solve($C, \Delta t$):

for all particles i of C
  compute $\Delta \mathbf{x}_i$
  $\mathbf{x}_i \leftarrow \mathbf{x}_i + \Delta \mathbf{x}_i$

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Iterations vs. Sub-steps

while simulating
  for all particles i
    $\mathbf{v}_i \leftarrow \mathbf{v}_i + \Delta t\ \mathbf{g}$
    $\mathbf{p}_i \leftarrow \mathbf{x}_i$
    $\mathbf{x}_i \leftarrow \mathbf{x}_i + \Delta t\ \mathbf{v}_i$
  for n iterations
    for all constraints C
      solve$(C, \Delta t)$
  for all particles i
    $\mathbf{v}_i \leftarrow (\mathbf{x}_i - \mathbf{p}_i)/\Delta t$

$\Delta t_s \leftarrow \Delta t / n$
while simulating
  for n substeps
    for all particles i
      $\mathbf{v}_i \leftarrow \mathbf{v}_i + \Delta t_s \mathbf{g}$
      $\mathbf{p}_i \leftarrow \mathbf{x}_i$
      $\mathbf{x}_i \leftarrow \mathbf{x}_i + \Delta t_s \mathbf{v}_i$
    for all constraints C
      solve$(C, \Delta t_s)$
    for all particles i
      $\mathbf{v}_i \leftarrow (\mathbf{x}_i - \mathbf{p}_i)/\Delta t_s$

- Spending fixed time budget with sub-steps is much is more effective than with iterations!
- XPBD much simpler (no λ tracking)

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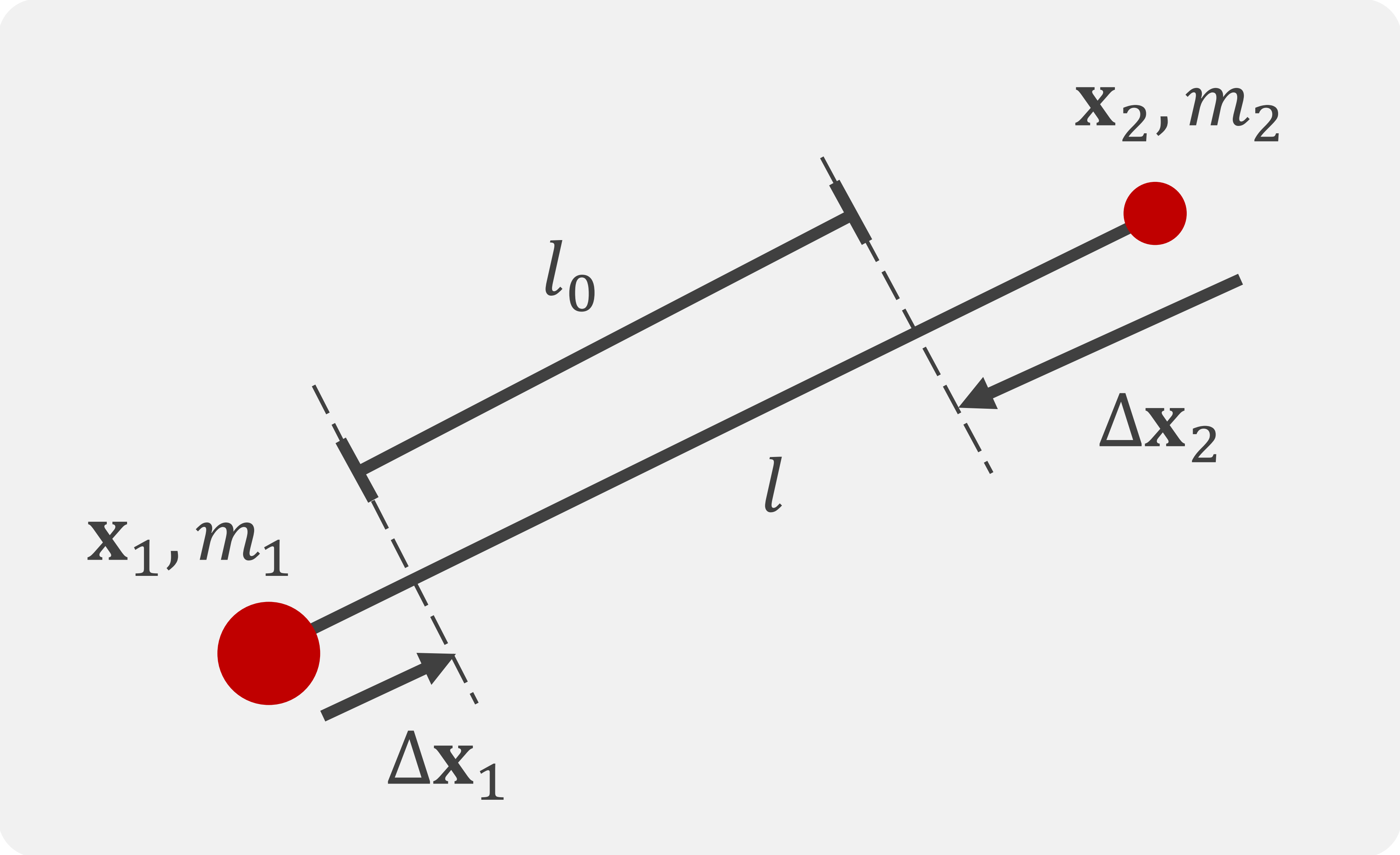
Distance Constraint

- Rest distance l_0
- Current distance l
- Masses m_i
- Inverse masses $w_i = 1/m_1$

$$\Delta \mathbf{x}_1 = \frac{w_1}{w_1 + w_2}(l - l_0)\frac{\mathbf{x}_2 - \mathbf{x}_1}{|\mathbf{x}_2 - \mathbf{x}_1|}$$

$$\Delta \mathbf{x}_2 = -\frac{w_2}{w_1 + w_2}(l - l_0)\frac{\mathbf{x}_2 - \mathbf{x}_1}{|\mathbf{x}_2 - \mathbf{x}_1|}$$



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General Constraint

$\mathbf{x}_1$

$\mathbf{x}_2$

…

$\mathbf{x}_n$

particles participating in constraint

$C(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$

constraint function

C

constraint error (number)

For the distance constraint:

$$C_{\text{dist}}(\mathbf{x}_1, \mathbf{x}_2) = |\mathbf{x}_2 - \mathbf{x}_1| - l_0$$

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Constraint Gradient ∇C

$\nabla C(\mathbf{x})$ is a vector

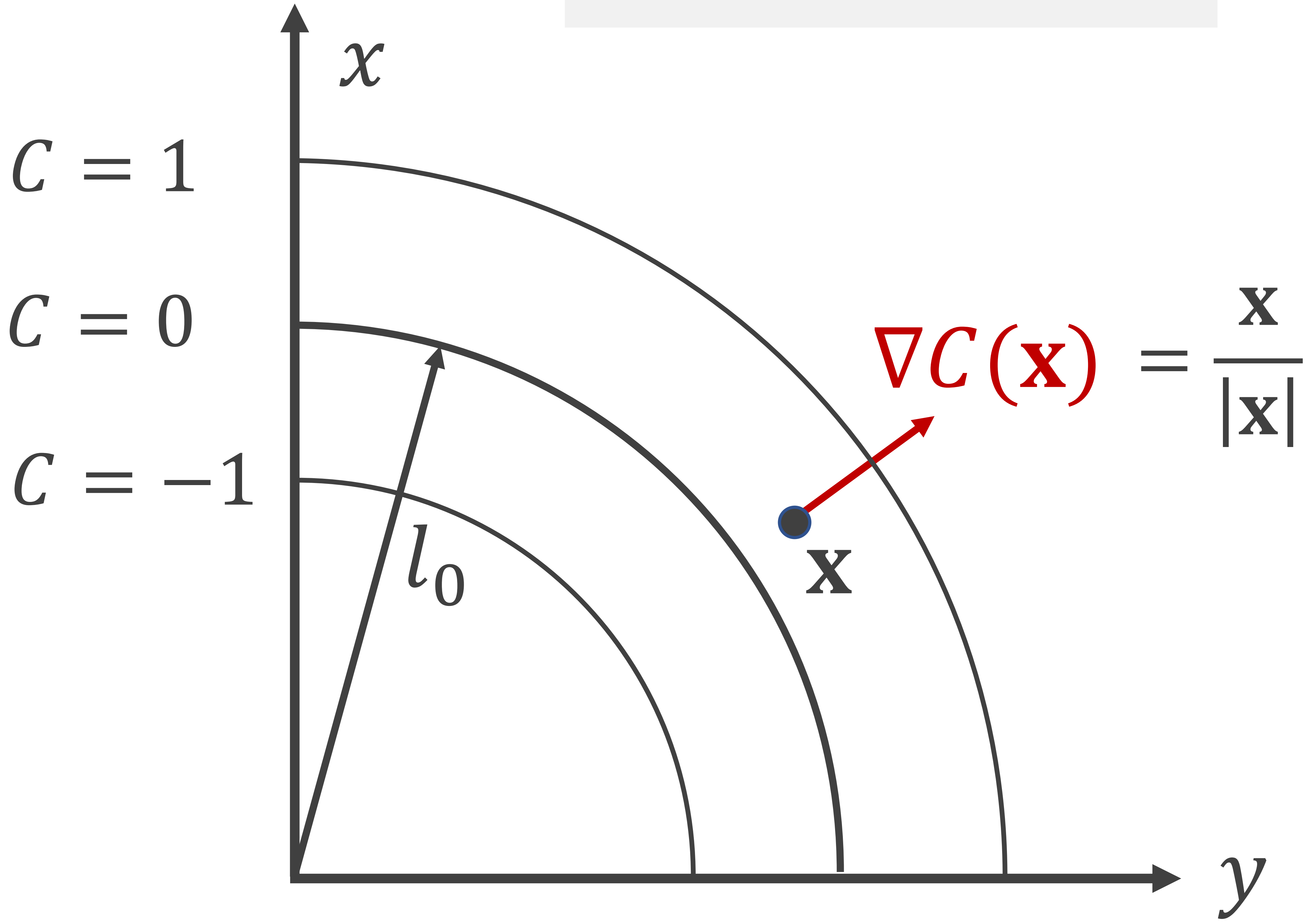
Vector direction:

- direction in which C increases the most

Vector length:

- how much C changes when moving $\mathbf{x}$ by one unit

$C_{\text{dist}} = |\mathbf{x}| - l_0$



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Solving a General Constraint (PBD)

Compute the scalar value λ (same for all participating particles):

$$\lambda = \frac{-\mathrm{C}}{w_1|\nabla C_1|^2 + w_2|\nabla C_2|^2 + \cdots + w_n|\nabla C_n|^2}$$

∇C_i: How to move $\mathbf{x}_i$ for a maximal increase of C

$|\nabla C_i|^2$ the squared length of ∇C_i

Compute correction for point $\mathbf{x}_i$ as: $\Delta\mathbf{x}_i = \lambda w_i \nabla C_i$

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Making the Constraint Soft

PBD:

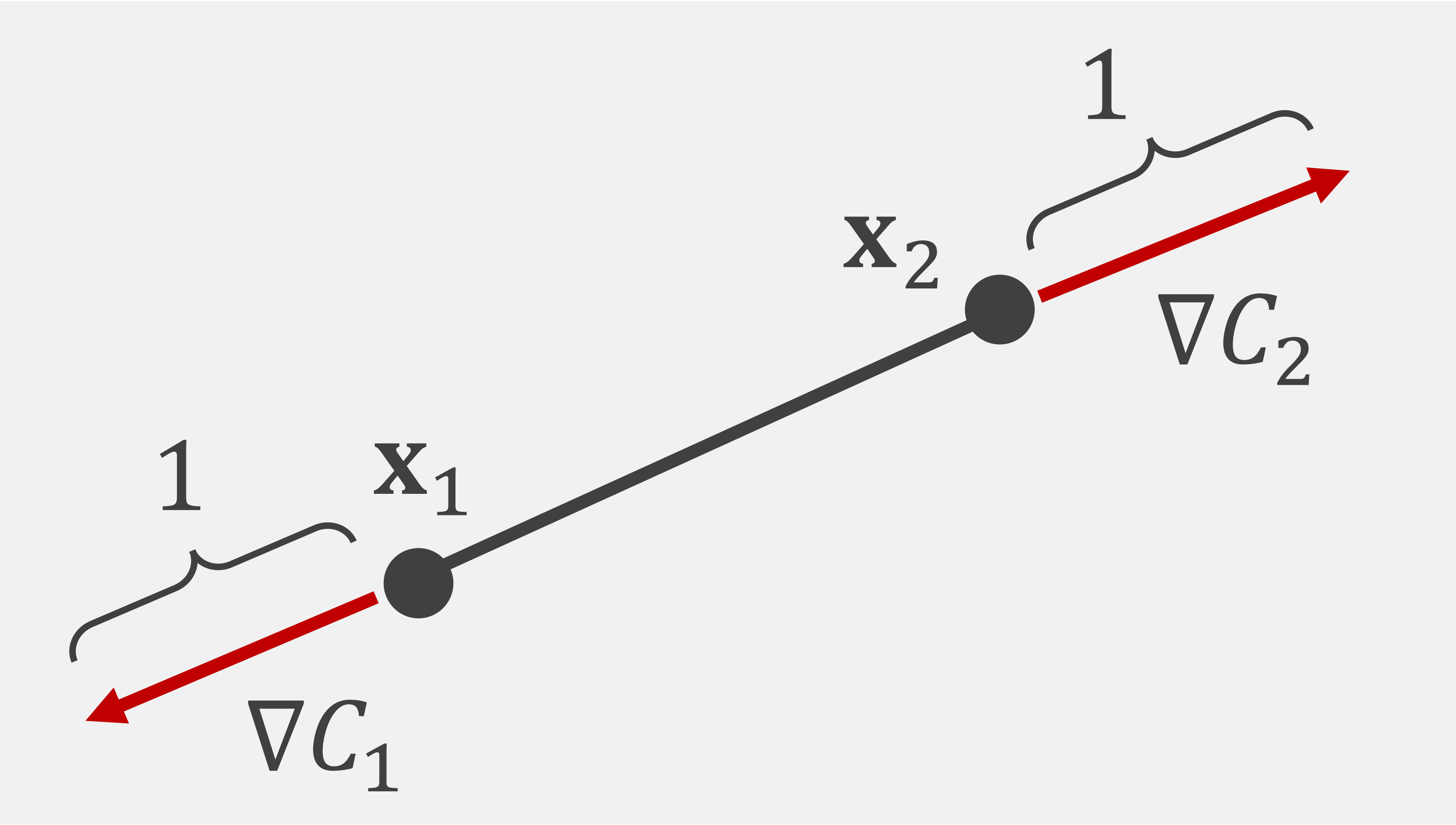
- Scale the correction as $\Delta \mathbf{x}_i = k\, \lambda w_i\, \nabla C_i$
- Stiffness $k \in [0,1]$
- Easy to tune!
- Dependent on time step (stiffer for smaller time steps)

XPBD:

$$\lambda = \frac{-C}{w_1|\nabla C_1|^2 + w_2|\nabla C_2|^2 + \cdots w_n|\nabla C_n|^2 + \frac{\alpha}{\Delta t^2}}$$

- Compliance α is the inverse of physical stiffness
- Infinitely stiff (hard) when $\alpha = 0$

Example: Distance Constraint



$$\nabla C_1 = \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|}$$

$$\nabla C_2 = \frac{\mathbf{x}_2 - \mathbf{x}_1}{|\mathbf{x}_2 - \mathbf{x}_1|}$$

$$\lambda = \frac{-\mathrm{C}}{w_1|\nabla C_1|^2 + w_2|\nabla C_2|^2 + \cdots + w_n|\nabla C_n|^2} = \frac{-(l - l_0)}{w_1 \cdot 1 + w_2 \cdot 1}$$

$$\Delta\mathbf{x}_1 = \lambda\, w_1\, \nabla C_1 = -\frac{w_1}{w_1 + w_2}(l - l_0)\frac{\mathbf{x}_2 - \mathbf{x}_1}{|\mathbf{x}_2 - \mathbf{x}_1|}$$

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