CS5630 Physically Based Realistic Rendering

Steve Marschner

Spring 2026

14 Path Tracing 3

Topics

Illumination from area lights as area integration

Area form of Rendering Equation

Path integral formulation of Rendering

Path tracing as path-space sampling

Operator form of Rendering Equation and Neumann expansion

Area lights

Suppose we have a surface point illuminated by an area light

- area light has radiance $L_s(y)$ at point y
- incident radiance field at x is $L_i(\omega_i) = L_s(\psi(x, \omega_i))$ when the ray tracing succeeds
- incident radiance is zero for other directions

Reflected radiance via surface reflection equation

- integral over ω_i
- may like to choose ω_i by sampling the source

Area lighting by area integration

The ray tracing function ψ creates a 1:1 relationship

- one side is a subset of the hemisphere, Ω_S
- the other side is a subset of the source, S_{vis}

Opportunity for a change of variable

$$L_r(\omega_r) = \int_{S_{\text{vis}}} f_r(x, \omega_r, \omega_i(y)) L_s(y) \, |\, \omega_i \cdot n(x) \,| \, \left| \frac{d\omega_i}{dy} \right| \, dy$$

$$L_r(\omega_r) = \int_{S_{\text{vis}}} f_r(x, \omega_r, \omega_i(y)) L_s(y) \underbrace{\frac{|\, \omega_i \cdot n(x) \,|\, |\, \omega_i \cdot n(y) \,|}{\|x - y\|^2}}_{\text{geometry factor}} \, dy$$

Estimators for area and solid angle integrals

visibility function

1 if x and y can see each other, else 0

To estimate this integral with Monte Carlo, sample $y \sim p_A(y)$

- first make the integral over all of S: $L_r(\omega_r) = \int_S f_r(x, \omega_r, \omega_i(y)) L_s(y) G(x, y) V(x, y)\, dy$
- then $\dfrac{f(y)}{p(y)} = \dfrac{f_r(\omega_r, \omega_i(y)) L_s(y) G(x, y) V(x, y)}{p_A(y)}$

geometry factor

$$G(x, y) = \frac{\cos\theta_x \cos\theta_y}{\|x - y\|^2}$$

If we estimate the solid angle integral instead

- $\dfrac{f(\omega_i)}{p(\omega_i)} = \dfrac{f_r(\omega_r, \omega_i) L_s(y(\omega_i)) V(x, y) \cos\theta_x}{p_A(y)\|x - y\|^2 / \cos\theta_y}$
- which is the same!
- the actual estimator (and therefore the code) does not depend on which integral we use to derive the estimator

Area form of rendering equation

Can carry this area integration thing farther…

First change notation to eliminate directions and ray tracing

- $L(x \to y) = L_o(x, \overrightarrow{xy}) = L_i(y, \overrightarrow{yx})$
- $L^0(x \to y) = L_e(x, \overrightarrow{xy})$
- $f_r(y \leftrightarrow x \leftrightarrow z) = f_r(x, \overrightarrow{xy}, \overrightarrow{xz})$

Then area-source illumination integral becomes

- $L(x \to z) = \int_S f_r(z \leftrightarrow x \leftrightarrow y) L(y \to x) G(x \leftrightarrow y) V(x \leftrightarrow y) dA(y)$
- no more explicit mappings between points and directions … all points instead
- no more ray tracing function … same L on both sides

To a surface based rendering equation

Assume closed scene, so all illumination is from surfaces

- union of all surfaces in the scene is called $\mathscr{M}$

Use area light illumination considering entire scene an area source

- $$L(x \to z) = \int_{\mathscr{M}} f_r(z \leftrightarrow x \leftrightarrow y) L(y \to x) G(x \leftrightarrow y) V(x \leftrightarrow y) dA(y)$$

Add in emitted light

- $$L(x \to z) = L^0(x \to z) + \int_{\mathscr{M}} f_r(z \leftrightarrow x \leftrightarrow y) L(y \to x) G(x \leftrightarrow y) V(x \leftrightarrow y) dA(y)$$
- now this is an integral equation for $L(x \to y)$.

To a path integral form (summary)

Merge with area form of measurement equation

Recursively expand integral in this form

- comes out more neatly because domains are all the same
- result is an infinite series of integrals

Interpret each term as an integral over paths

- name the integrand as the contribution function

Merge domains into a single path space

Measurement equation in solid angle form

The measurement equation we wrote down in the cameras lecture:

- $$J_j = \frac{1}{f^2} \int_{\Delta t} \int_{R^2} \int_A L_i(\mathbf{y}, \omega(\mathbf{x}, \mathbf{y}))\, h(\mathbf{x} - \mathbf{x}_j)\ \cos^4 \theta\, d\mathbf{y}\, d\mathbf{x}\, dt$$
- this is an integral over points $\mathbf{y}$ on the aperture and points $\mathbf{x}$ on the image plane
- omit t (we can add it back later) and change variable from $\mathbf{x}$ to ω
- $$\Phi_j = \frac{1}{f^2} \int_{H^2} \int_A L(\mathbf{y}, \omega)\, h(\mathbf{x}(\omega) - \mathbf{x}_j)\ \cos^4 \theta \left| \frac{d\mathbf{x}}{d\omega} \right| d\mathbf{y}\, d\omega$$
- now integrating radiance over area on the aperture and solid angle looking into the scene
- summarize by introducing a weighting function $W(\mathbf{y}, \omega)$
- $\Phi_j = \int_A \int_{H^2} W_j(\mathbf{y}, \omega) L(\mathbf{y}, \omega) \cos \theta_y\, d\omega\, d\mathbf{y}$ — inner integral looks just like surface reflection

Measurement equation in area form

Make analogous conversion to the area lighting one

- $\Phi_j = \int_A \int_{H^2} W_j(\mathbf{y}, \omega) L(\mathbf{y}, \omega) \cos\theta_y \, d\omega \, d\mathbf{y}$

compare to area-light illumination equation:

$L_r(\omega_r) = \int_S f_r(x, \omega_r, \omega_i(y)) L_s(y) G(x, y) V(x, y) \, dy$

$W = 0$ for backwards ω

- $\Phi_j = \int_A \int_{\mathscr{M}} W_j(\mathbf{y}, \omega(\mathbf{x})) L(\mathbf{y}, \omega(\mathbf{x})) G(\mathbf{x}, \mathbf{y}) V(\mathbf{x}, \mathbf{y}) \, d\mathbf{x} \, d\mathbf{y}$

dropping bold has no significance!
(just saves some space)

- $\Phi_j = \int_A \int_{\mathscr{M}} W_j(y \leftarrow x) L(y \leftarrow x) G(x \leftrightarrow y) V(x \leftrightarrow y) dA(x) dA(y)$
- this is now the area-form measurement equation
- flux for j-th pixel as a weighted integral of L

Recursive expansion

Let's name:

- the aperture point: x_0
- the visible surface point: x_1
- the product $G(x \leftrightarrow y)V(x \leftrightarrow y)$: $\bar{G}(x \leftrightarrow y)$

Then:

- $\Phi_j = \int_A \int_{\mathscr{M}} W_j(x_0 \leftarrow x_1)L(x_0 \leftarrow x_1)\bar{G}(x_0 \leftrightarrow x_1)dA^2(x_0x_1)$
- $L(x_0 \leftarrow x_1) = L^0(x_0 \leftarrow x_1) + \int_{\mathscr{M}} f_r(x_0 \leftrightarrow x_1 \leftrightarrow x_2)L(x_2 \leftarrow x_1)\bar{G}(x_1 \leftrightarrow x_2)dA(x_2)$
- substitute on board…

Recursive expansion

$$
\begin{aligned}
\Phi_j &= \int_A \int_{\mathscr{M}} W_j(x_0 \leftarrow x_1) L(x_0 \leftarrow x_1) \bar{G}(x_0 \leftrightarrow x_1) dA^2(x_0 x_1) \\
&= \int_A \int_{\mathscr{M}} W_j(x_0 \leftarrow x_1) \left[L^0(x_0 \leftarrow x_1) + \int_{\mathscr{M}} f_r(x_0 \leftrightarrow x_1 \leftrightarrow x_2) L(x_2 \leftarrow x_1) \bar{G}(x_1 \leftrightarrow x_2) dA(x_2) \right] \bar{G}(x_0 \leftrightarrow x_1) dA^2(x_0 x_1) \\
&= \int_A \int_{\mathscr{M}} W_j(x_0 \leftarrow x_1) \bar{G}(x_0 \leftrightarrow x_1) L^0(x_0 \leftarrow x_1) dA^2(x_0 x_1) \\
&\quad + \int_A \int_{\mathscr{M}^2} W_j(x_0 \leftarrow x_1) \bar{G}(x_0 \leftrightarrow x_1) f_r(x_0 \leftrightarrow x_1 \leftrightarrow x_2) \bar{G}(x_1 \leftrightarrow x_2) L^0(x_2 \leftarrow x_1) dA^3(x_0 x_1 x_2) \\
&\quad + \cdots
\end{aligned}
$$

Path integral

To neaten this up define

- A is part of $\mathscr{M}$, and $W_j(x \to y) = 0$ when y is not on the aperture
- $\bar{\mathbf{x}}$ can stand for any sequence $x_0 x_1 \cdots x_n$ — this is a *path*
- $\mathscr{P} = \cup_{n=2}^{\infty} \mathscr{M}^n$ — this is the space of all paths, or *path space*
- $d\bar{A}(\bar{\mathbf{x}})$ is $dA^n(x_0 x_1 \cdots x_n)$ — *area product measure*, automatically the right dimension for the path

Then we can summarize everything

$$\Phi_j = \int_{\mathscr{P}} f_j(\bar{\mathbf{x}}) d\bar{A}(\bar{\mathbf{x}})$$

where $$f_j(x_0 \ldots x_n) = W_j(x_0 \leftarrow x_1) G(x_0 \leftrightarrow x_1) \left[\prod_{k=1}^{n-1} f_r(x_{k-1} \leftrightarrow x_k \leftrightarrow x_{k+1}) G(x_k \leftrightarrow x_{k+1}) \right] L_0(x_{n-1} \leftrightarrow x_n)$$

Monte Carlo path integration

This summarized form permits very high level thinking

- Integral is $I = \int f(\bar{\mathbf{x}}) d\bar{A}(\bar{\mathbf{x}})$
- suppose we generate a path $\bar{\mathbf{x}}_i \sim p(\bar{\mathbf{x}}_i)$
- estimator is then $g(\bar{\mathbf{x}}_i) = \dfrac{f(\bar{\mathbf{x}}_i)}{p(\bar{\mathbf{x}}_i)}$

This is the starting point for many advanced path tracing methods

- but we can start by looking at our familiar path tracer through this lens

Path tracing as an importance sampling scheme

Consider the decisions in tracing a path using BRDF sampling only

1. choose x_0 on the camera aperture: $x_0 \sim p_A$ (area)
2. choose ω_0 to sample the pixel footprint: $\omega_0 \sim p_W$ (projected solid angle)
3. trace a ray and call the intersection x_1: $p(x_1 \,|\, x_0) = p_W(\omega_0)\bar{G}(x_0 \leftrightarrow x_1)$ (area)

- now we have sampled a path of length 2 with density

$$p(x_0 x_1) = p(x_0)p(x_1 \,|\, x_0) = p_A(x_0)p_W(\omega_0)\bar{G}(x_0 \leftrightarrow x_1)$$

$$p_W(x_0 \leftarrow x_1) = p_A(x_0)p_W(\overrightarrow{x_0 x_1})$$

$$g(\bar{\mathbf{x}}) = \frac{f(\bar{\mathbf{x}})}{p(\bar{\mathbf{x}})} = \frac{W(x_0 \leftarrow x_1)\bar{G}(x_0 \leftrightarrow x_1)L^0(x_0 \leftarrow x_1)}{p_A(x_0)p_W(\omega_0)\bar{G}(x_0 \leftrightarrow x_1)} = \frac{W(x_0 \leftarrow x_1)}{p_W(x_0 \leftarrow x_1)} \frac{\bar{G}(x_0 \leftrightarrow x_1)}{\bar{G}(x_0 \leftrightarrow x_1)} L^0(x_0 \leftarrow x_1)$$

- the first ratio is constant, since p_A and p_W are exactly designed to importance sample W

Path tracing as an importance sampling scheme

Continuing from the first intersection…

4. select direction ω_1 according to BRDF at x_1: $p(\omega_1 \,|\, x_0 x_1) = p_r(\omega_1 \,|\, x_1, \overrightarrow{x_1 x_0}) \approx C f(x_1, \overrightarrow{x_1 x_0}, \omega_1)$
5. ray trace and call the intersection x_2: $p(x_2 \,|\, x_0 x_1) = p_r(x_0 \leftrightarrow x_1 \leftrightarrow x_2)\bar{G}(x_1 \leftrightarrow x_2)$

- now we have sampled a path of length 3 with density:

$$p(x_0 x_1 x_2) = p(x_0)p(x_1 \,|\, x_0)p(x_2 \,|\, x_0 x_1) \propto W(x_0 \leftarrow x_1)\bar{G}(x_0 \leftrightarrow x_1)f_r(x_0 \leftrightarrow x_1 \leftrightarrow x_2)\bar{G}(x_1 \leftrightarrow x_2)$$

- and the estimator reads:

$$g(\bar{\mathbf{x}}) = \frac{f(\bar{\mathbf{x}})}{p(\bar{\mathbf{x}})} = \frac{W(01)\bar{G}(01)f_r(012)\bar{G}(12)L^0(12)}{p_W(01)\bar{G}(01)p_r(012)\bar{G}(12)} = \frac{W(01)}{p_W(01)}\frac{f_r(012)}{p_r(012)}L^0(x_1 \leftarrow x_2)$$

where (01) stands for $(x_0 \leftarrow x_1)$, etc.

- again we've approximately canceled everything but L^0

Longer paths, RR, MIS

This process continues to many bounces

- generates one path of each length, aka. one sample from each piece of path space
- these samples are used to estimate the integrals in the infinite sum

With Russian Roulette

- this simply scales down the probability densities for longer paths
- corrected by dividing by the product of the survival probabilities

With MIS between sampling lights and sampling BRDF

- we generate two samples of each path length
- MIS weights correctly combine these to remain unbiased

Operator form of RE (on board)

Summarize solid-angle form in terms of two integral operators

- scattering operator
- propagation operator

Brings RE into a simple abstract form

- solve using Neumann series (analogous to series for $1/(1-x)$)
- provides alternate explanation for infinite series of integrals