CS5630 Physically Based Realistic Rendering

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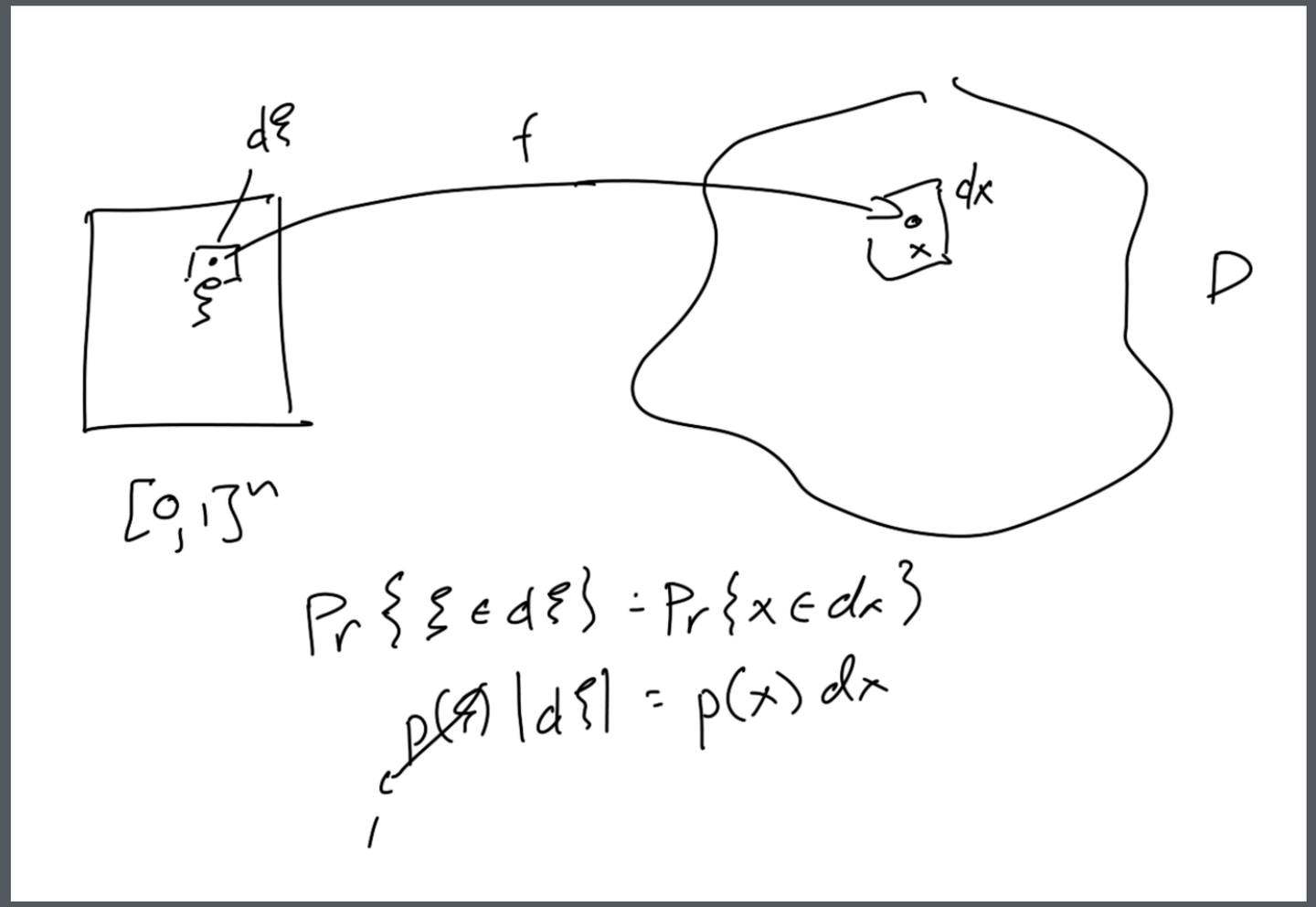
05 Sampling techniques

Choosing random values

More general view in n-D

- sample generator takes in n random numbers
  (aka. a point ξ in the unit n-D hypercube)
- then computes a continuous bijective function $x = f(\xi)$
- neighborhood $d\xi$ maps to neighborhood dx
- probabilities are the same
- then $p(x) = \left| \frac{dx}{d\xi} \right|^{-1}$



Choosing random values

In code random sampling amounts to implementing $f()$ from the previous slide

Input is an n-D point; output is an n-D point

- e.g. `Point2 sample(Point2 seed)`
- passing ξ explicitly allows use of stratified, etc. patterns

Because it implements a continuous 1:1 map we call this a warping function.

Examples of this general pattern

Sampling an analytic distribution in 1D

Uniformly sampling a rectangle in 3D

Uniformly sampling a disc (nbdemo)

Uniformly sampling the surface of a sphere (nbdemo)

Sampling a tabulated distribution in 1D

Sampling a Gaussian distribution with Box-Muller

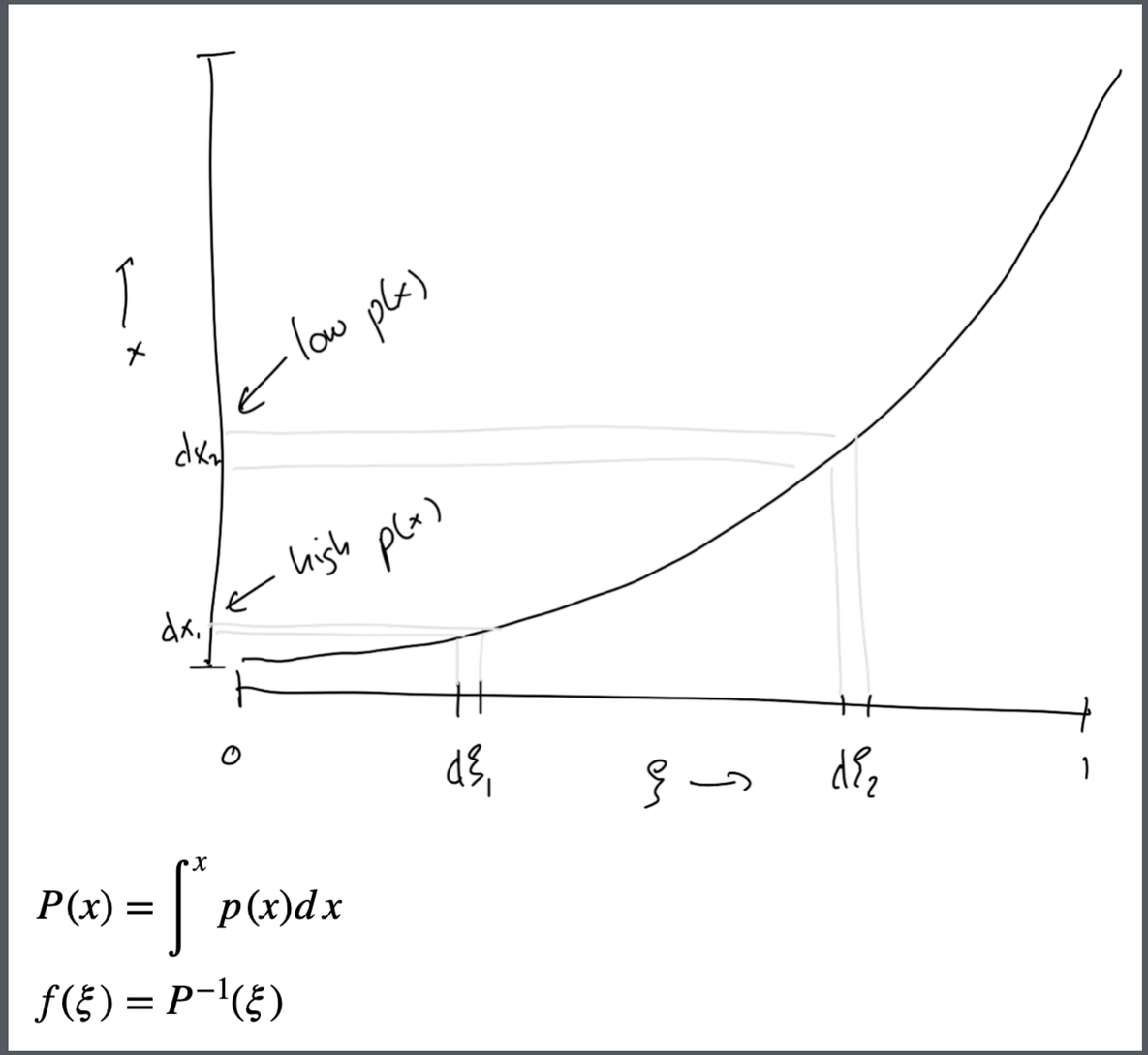
1D warp: Inverse CDF redux

In 1D:

- $p(f(\xi)) = \dfrac{1}{|f'(\xi)|}$ or $p(x) = |(f^{-1})'(x)|$
- if we set f^{-1} to the function
  $P(a) = \int_0^a p(x)\,dx$ (the CDF of p)
  then $(f^{-1})'(x) = p(x)$
- so we want f to be the inverse of P

This is the same thing we saw last time

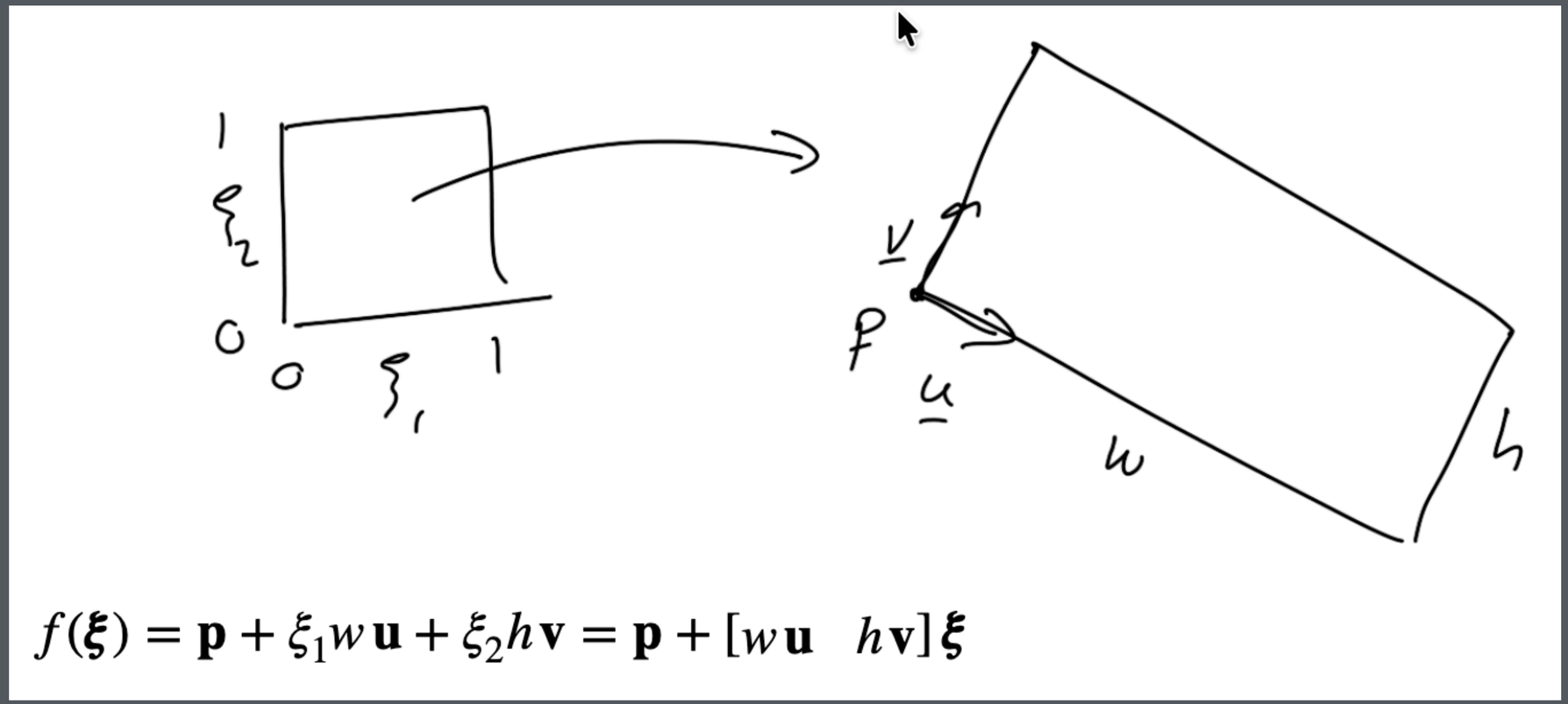
- then by matching CDFs
- this time by designing the warp to have the right derivative



Sampling a rectangle in 3D

Suppose f is a linear map $f(x) = Ax + b$

- if A is 3x2 then this maps the unit square to a rectangle in 3D space
- what is the pdf?



$$f(\boldsymbol{\xi}) = \mathbf{p} + \xi_1 w\,\mathbf{u} + \xi_2 h\,\mathbf{v} = \mathbf{p} + [w\mathbf{u} \quad h\mathbf{v}]\,\boldsymbol{\xi}$$

Sampling in polar coordinates

Many problems in rendering involve circularly symmetric domains

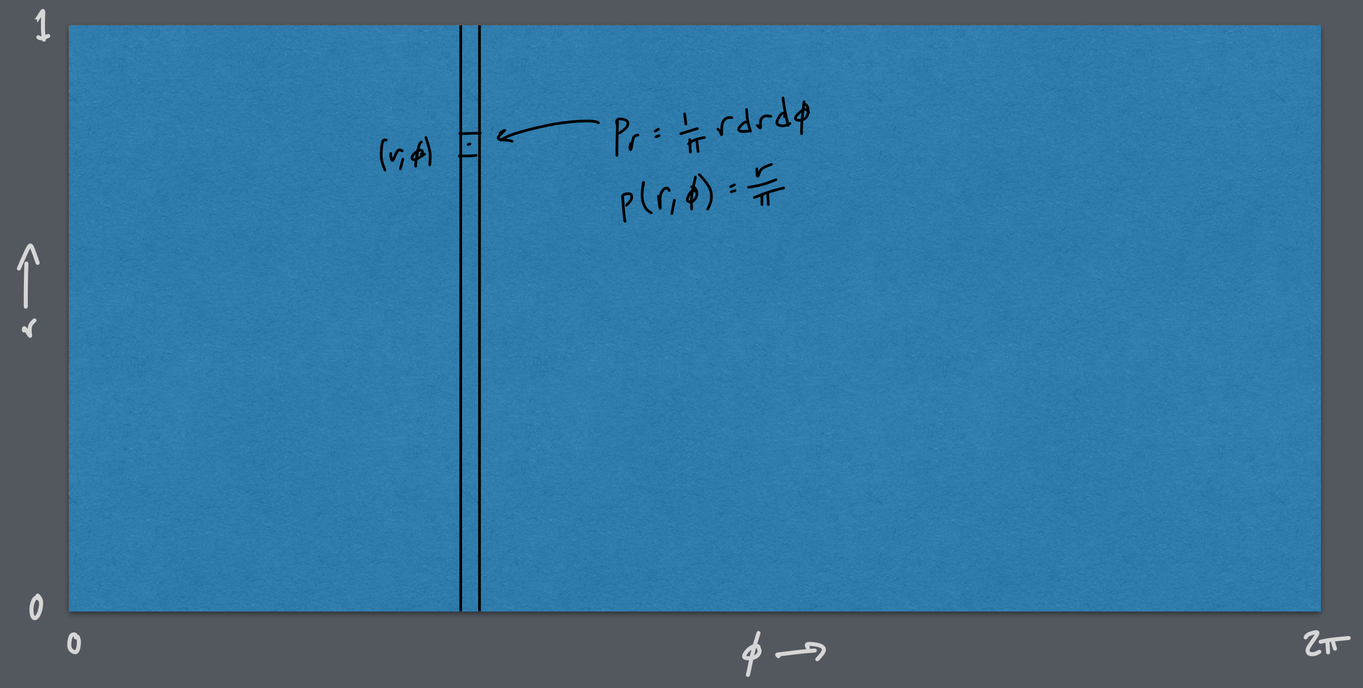
- discs
- hemispheres
- spheres

Polar coordinates are a nice tool for this

- viewed in the (r, ϕ) domain a disc is a rectangle!
- we know how to sample a rectangle…will it work?
- (notebook)

Polar coordinates warp

What does a uniform density in the disc look like in the (r, ϕ) rectangle?



- In general, to achieve a desired distribution $p(x)$ we want $p(r, \phi) = rp(x)$

Polar coordinates sampling applications

Disc

- (notebook)

Hemisphere

- (notebook)

Hemisphere with $p = \cos\theta$

- (notebook)

Box-Muller sampling algorithm

Sampling components of shading models

- very common application; see next assignment!

Statistical software testing

How to test a program whose output is by design uncertain?

- use statistics of course!

Testing values

- want to write “assert(function(x) == y)”, but only the expected value is guaranteed
- similar to experimental science where we make measurements and ask whether they match the value predicted by some theory
- same tools apply

Most basic tool is Student’s T test (one-sample version)

- input: many samples from some distribution, assumed Gaussian
- output: probability of sample mean being as large as it is if the mean were zero
- accept that “null hypothesis” if the probability is above some threshold, declare test passed

Statistical software testing

Testing distributions

- want to answer “does y = f() return samples distributed according to p(y)?”
- can answer a discrete version of this question

Another classical statistics tool: the χ^2 test

- input: histogram of a set of samples and reference probabilities for each bin
- output: probability that the observed sample counts would differ this much from the expected numbers if the reference probabilities were true
- accept null hypothesis if the probability is above some threshold, declare test passed

For continuous distributions, test by cutting domain into bins

- integrate pdf over each bin to get probability
- this is how Nori’s statistical tests work