CS 522 Assignment 4 May 3, 2001

The assignment should be done alone; however, discussions with fellow students are encouraged. Matlab programs should be sufficiently documented. **For this assignment, you will need to use your Matlab functions from Assignment 1 and 2. Make sure your functions are correct before doing Assignment 4**.

1. (25 Points) Assume that the underlying asset follows a geometric Brownian motion, i.e.,

$$dS_t = \mu S_t dt + \sigma S_t dX_t$$

where $\mu, \sigma > 0$, and X_t is a standard Brownian motion. In addition, the underlying asset pays a discrete dividend qS_{t_d} at time $0 \leq t_d \leq T$ where $q > 0$. By modifying your functions **MyBSPrice** and **BinomialPrice** in Assignment 1, write the Matlab functions

MyBinPrice(S0,E,T,r,σ,M,q,t_d,flag,flagA)

and

BinPrice($S0, E, T, r, u, d, M, q, t_d$, flag,flagA)

so that they return the initial **American** option value for this underlying asset, which pays a single dividend with the yield $q > 0$ at t_d when flagA =1 and the initial **European** option value when flagA =0. (Note that, for computational time efficiency, the tree should be built before computing the option values and the dividend payment affects only the tree construction phase of the binomial method.)

Let $S_0 = 100$, $r = 4\%$, $\sigma = 30\%$, $M = 250$, and $T = 1$.

(a) In one plot, graph $\max(S_0 - E, 0)$ as a function of the strike E and the initial **European** call option price $C(E, T)$, assuming underlying pays no dividend payment, for strike price $E = 10 : 5 : 120$;

(b) Now let $q = 2\%$ and $t_d = 0.5$. Graph $\max(S_0 - E, 0)$ as a function of the strike E and the initial **European** call option price $C(E, T)$ for strike price $E = 10 : 5 : 120$;

(c) Using the same parameter setting as (b), now plot the initial **American** call option value (when underlying pays a single dividend payment qS at t_d) and the function $\max(S_0 - E, 0)$ as a function of the strike E for strike price $E = 10 : 5 : 120$.

Comment on your observations of the plots in (a), (b), and (c).

2. (30 Points) Assume that the interest rate $r > 0$ is constant and the underlying asset price follows an absolute diffusion, i.e.,

$$\frac{dS_t}{S_t} = (\mu - q)dt + \frac{C}{S}dX_t$$

where $\mu > 0$, $q \geq 0$ (continuous dividend yield), $\sigma(S,t) = \frac{C}{S}$ is the local volatility function, and X_t is a standard Brownian motion.

(a) (5 Points) Add an input argument **flag** to your Matlab function **FwdBSEuroPut** in Assignment 2 so that it returns the call option value when **flag** = 1 and put option value when **flag** =0 by directly modifying initial and boundary conditions (call the new function **FwdBSEuro**).

Assume that the parameters are specified as follows:

```
S0=370, r = 0.1,  q = 0,  C = 30, T = Tmax = 1,
Emax = 2*S0, N = 200, M = 100.
```

You may set $\sigma(0)$ arbitrarily equal 100 as in Assignment 2.

Output initial call option values corresponding to $T = 1$ and the (grid) strike price in the range $[340, 410]$.

(b) (15 Points) Modify the matlab function in the midterm **EuroCall** (You can download it from the course web) so that it generates the price of a down-and-out put option using the **implicit finite difference** method for the **backward** Black-Scholes PDE, i.e., complete the following **Matlab** function

```
function Val = DownOutPut(sigma,r,q,Smax,Xd,E,T,N,M)
%
% Pre
%   sigma :    local volatility at grid points
%   r :        positive constant interest rate
%   q :        positive constant continuous dividend yield
%   Smax:      a large positive number
%   Xd:        down barrier
%   T :        expiry date
%
% Post
%   Val:       the initial Down-and-Out Put option values
%
```

Using the same parameters in (a) except now $S_{\max} = 2S0$, $q = 2\%$, $E = S_0$, and $X_d = 340$, output initial **down-and-out put** option values corresponding to the (grid) initial price in the range $[340, 410]$.

(c) (10 Points) Use the same parameters as (b) but assume now q=0. Compute the **implied volatility** of an option with strike E and expiry T using the Matlab

function **BLSIMPV** by first calculating the call option price using your **FwdBSEuro**. For each (grid) strike price $E_i = i\frac{2S_0}{N}$ in the range $[S0, 1.02S_0]$, graph in one plot the initial **down-and-out put** values using the implied volatility of the corresponding European call with the strike E_i and the local volatility function $\frac{C}{S}$ against the initial underlying prices. Comment on the accuracy of the down-and-out put price when the implied volatility, instead of the given local volatility function, is used.

3. (10 Points) A repeated hit in-barrier call option has barriers at levels X_u and X_d above and below the initial price S_0 respectively. If the asset price S_t touches both barriers before the expiry T, the option has payoff $\max(S - E, 0)$. Otherwise the option does not pay off. Let $V(S, t)$ denote the value of this option at time t and asset price S, assuming that neither price level has been hit before time t. Formulate this barrier option pricing problem as partial differential equations with suitable boundary and final conditions; you need to describe all the PDE equations and their boundary and final conditions used. (Hint: the first time that one of the price levels is hit, the contract becomes a barrier contract with a single price level).

4. (15 Points) Consider a coupon bond that pays C_t dollars at times $t = 1, 2, \cdots, T$. The last payment C_T includes the principal repayment. Let $V(t)$ represent the price of the coupon bond at time t. Let $P(t, T)$ represent the time t value of the zero-coupon bond which pays 1 dollar at time T.

   Assume that there is no arbitrage (and no possibility of default). Without making any assumption on the interest rate model, show that

$$V(0) = \sum_{t=1}^{T} C_t P(0, t).$$

5. (20 Points) In the first continuous interest rate model, Vasicek (1977) assumes that the instantaneous short rate satisfies

$$dr_t = \kappa(\mu - r_t)dt + \sigma dX_t \tag{1}$$

   where $\kappa, \mu, \sigma > 0$ are constants, and X_t is a standard Brownian motion. Let λ be a constant denoting the market price of risk.

   (a) What is the partial differential equation satisfied by the zero coupon bond price $P(t, T)$?

   (b) By solving the PDE problem analytically, it can be shown that the zero-coupon bond price is

$$P(t, T) = e^{A(t,T) - B(t,T)r_t}$$

where

$$A(t,T) = (B(t,T) - T + t)R_\infty - \tfrac{\sigma^2}{4\kappa}B^2(t,T)$$
$$B(t,T) = \tfrac{1-e^{-\kappa(T-t)}}{\kappa}$$
$$R_\infty = \mu - \tfrac{\lambda\sigma}{\kappa} - \tfrac{1}{2}(\tfrac{\sigma}{\kappa})^2$$

Show that the yield curve is given by

$$R(t,T) = R_\infty + \frac{B(t,T)}{T-t}(r_t - R_\infty) + \frac{\sigma^2 B(t,T)^2}{4\kappa(T-t)}.$$

What is $\lim_{T\to+\infty} R(t,T)$?

(c) In one plot, for $\kappa \in \{5, 0.5, 0.01\}$, graph yield curves $R(t,T)$ for $T \in (t, t+30]$ with $\sigma = 0.01$, $\lambda = -0.2$, $r_t = 0.03$ and $\mu = 0.05$. Comment on the effect of the parameter κ on the shape of the yield curve.