CS 522 **Assignment 3** **April 5, 2001**

The assignment should be done alone; however, discussions with fellow students are encouraged. Matlab programs should be sufficiently documented.

1. (15 Points) Assume that the underlying asset price S_t follows a geometric Brownian motion and there is no dividend payment. Then, at time t when underlying price is S, the value of a **binary** option, which pays \$1 at the expiry T if $S_T > E$ and otherwise zero, can be computed using the analytic formula

$$B(S,t) = e^{-r(T-t)}N(d_2)$$

where $0 \leq t \leq T$ and $N(d)$ is the cumulative distribution function of a standard normal and d_2 is the same as in the Black-Scholes formula. Consider the **pay-later** call option. This option has payoff $\max(S-E,0)$ at the expiry T. However, the holder of the option does not pay a premium when the contract is set up but must pay Q to the writer at expiry only if $S \geq E$. Determine the value of Q using the Black-Scholes formula for a European call and the analytic formula for the binary option.

2. (15 Pionts) Assume that the stock S_0 goes to either uS_0 or dS_0 at the expiry T and $u > e^{rT} > d$ where $r > 0$ is the risk-free interest rate (a 1-period binomial model). In addition the stock pays no dividend. Show that it is never optimal to exercise an American call option on this stock early.

3. (15 Points) Assume that the asset S pays no dividend and there is no arbitrage. Let C_t^A and P_t^A denote the time t values of an American call and put on the underlying asset S_t with the strike E and expiry T. Using the fact that it is not optimal to exercise a call option early when there is no dividend, show the following **put-call inequalities** for American options

$$S_t - E \leq C_t^A - P_t^A \leq S_t - Ee^{-r(T-t)}.$$

4. Assume that the asset price follows a geometric Brownian motion

$$\frac{dS_t}{S_t} = (\mu - q)dt + \sigma dX_t$$

where μ, q, σ are positive constants, and X_t is a standard Brownian motion. Assume that $r > q > 0$ where r the the constant risk free interest rate. At time t when the underlying has price S, let $V(S,t)$ denote the American option value with a payoff function payoff(S).

(a) (10 Points) Verify that under the transformation

$$S = Ee^x, \quad t = T - \frac{2\tau}{\sigma^2}, \quad V = Ee^{\alpha x + \beta\tau} u(x, \tau),$$

where $\alpha = \frac{1-k_1}{2}$, $\beta = -(\frac{(k_1-1)^2}{4} + k_2)$, $k_1 = \frac{2(r-q)}{\sigma^2}$, and $k_2 = \frac{2r}{\sigma^2}$, the partial differential complementarity formulation for an American option becomes:

$$\begin{cases} \frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \geq 0, \ u \geq g(x, \tau) \\ (u - g(x, \tau))(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}) = 0 \end{cases} \tag{1}$$

for $-\infty < x < \infty$, $0 \leq \tau < \frac{1}{2}T\sigma^2$, with $u(x, 0) = g(x, 0)$ and, for a put,

$$g(x, \tau) = e^{-(\alpha x + \beta \tau)} \max(1 - e^x, 0).$$

(b) (10 Points) Consider a uniform discretization in x and τ:

$$\{(x_i, \tau_j)\}_{i=0,\cdots,N}^{j=0,\cdots,M} \approx [x_{\min}, x_{\max}] \times [0, \tau_T]$$

where $\tau_T = \frac{1}{2}T\sigma^2$, and, for $i = 0, \cdots, N$, $j = 0, \cdots, M$,

$$x_i = x_{\min} + i\delta x, \ \tau_j = j\delta\tau, \ \delta x = \frac{x_{\max} - x_{\min}}{N}, \ \text{and } \delta\tau = \frac{\tau_T}{M}.$$

Let $z^j = [u_1^j; \cdots; u_{N-1}^j]$ and u_i^j denote the value u at τ_j and x_i, $j = 1, \cdots, M+1$, $i = 0, \cdots, N$. Using implicit finite differences, show that the discretized problem for (1) at τ_j is

$$\begin{cases} z^j \geq g^j, \quad Az^j - b^{j-1} \geq 0 \\ (z^j - g^j).*(Az^j - b^{j-1}) = 0 \end{cases} \tag{2}$$

where

$$A = \begin{bmatrix} 1+2\gamma & -\gamma & 0 & \cdots & 0 \\ -\gamma & 1+2\gamma & -\gamma & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -\gamma & 1+2\gamma & -\gamma \\ 0 & \cdots & 0 & -\gamma & 1+2\gamma \end{bmatrix}$$

with

$$b^{j-1} = \left[u_1^{j-1} + \gamma u_0^j; u_2^{j-1}; \cdots; u_{N-2}^{j-1}; u_{N-1}^{j-1} + \gamma u_N^j\right]$$
$$g^j = [g_1^j; \cdots; g_{N-1}^j], \ g_i^j = g(x_i, \tau_j), \ i = 1, \cdots, N-1$$

with $\gamma = \frac{\delta\tau}{\delta x^2}$. Verify that suitable boundary conditions for the American **put** option are $u_N^j = 0$ and $u_0^j = g(0, \tau_j)$.

(c) (25 Points) Using the **implicit** finite difference method, the **Brennan and Schwartz** method solves the discretized complementarity problem (2) at each time τ_j with a direct method. For simplicity, let us omit the superscript j for the time dependence, i.e., $z \equiv z^j$, $g \equiv g^j$, and $b \equiv b^{j-1}$. Let us left multiply A by an upper bidiagonal matrix U^{-1} with a unit diagonal to transform A into a lower bidiagonal matrix:

$$Az = b \qquad \Rightarrow \qquad Lz = U^{-1}b.$$

Then denote $y = U^{-1}g$ and solve a simpler lower bidiagonal complementarity problem:

$$\begin{cases} z \geq g, \quad Lz - y \geq 0 \\ (z-g).*(Lz-y) = 0 \end{cases} \tag{3}$$

Let $L = \text{diag}(l) + \text{diag}(e(2:n), -1)$ where n is dimension of the problem (2). It can be shown that, for a put option, the unique solution of (2) is the unique solution of (3) which can be solved by simply modifying the forward substitution method for $Lz = y$:

$$\begin{array}{ll} z_1 & = y_1/l_1 \\ \text{for} & i = 2:n \\ & z_i = (y_i - e_i * z_{i-1})/l_i; \\ & z_i = \max(z_i, g_i) \\ \text{end} & \end{array}$$

You are given three **Matlab** functions **TriDiUL**, **UUBidiSol** and **LLBidiSol**; you can download them from the course web. By modifying $x =$ **LLBidiSol**(e, l, y), complete the Matlab function below:

```
function x = LLBidiSolCP(e,l,y,g)
%
% Pre:
%    l,e  n-vector that defines the diagonal and lower diagonal of L
%         L = diag(l) + diag(e(2:n),-1).
%    y    n-vector
%
% Post:
%    x    n-vector that solves Lx >= y  x>=g and (x-g).*(Lx-y)=0
%
```

Using **TriDiUL**, **UUBidiSol** and **LLBidiSolCP**, write a **Matlab** function

$$V = \textbf{AmeriPutBS}(\sigma, r, q, E, T, x_{\min}, x_{\max}, N, M)$$

to compute the American put values using the Brennan and Schwartz method; here V is a $(N+1)$-vector with V_i equal to the computed option premium when underlying price is $Ee^{x_{\min}+(i-1)\delta x}$ and $t = 0$. Make your Matlab function as **efficient** as possible.

(d) (10 Points) Using the following parameters,

$$S_0 = 100, \ \sigma = 0.30, \ r = 0.1, \ q = 0.02, \ E = 100, \ T = 1,$$
$$x_{\min} = -\log(2S_0E), \ x_{\max} = \log(\tfrac{2S_0}{E}), \ M = 100, \ N = 500,$$

compute American put premiums using **AmeriPutBS**.

i. Print out the option premiums corresponding to the discretized underlying prices in the interval $[.9S_0, 1.1S_0]$. Graph the premium computed against the initial asset price at each grid point $Ee^{x_{\min}+(i-1)\delta x}$, $i = 1, 2, \cdots, N+1$.

ii. Using forward finite difference, compute the delta hedge factor

$$\Delta_i \approx \frac{V_{i+1} - V_i}{\delta S_i}, \quad i = 0, \cdots N-1$$

with appropriate δS_i. Graph the delta values Δ computed from using **AmeriPutBS** against the corresponding asset price.

Bonus. (10 Points) Modify your Matlab program so that it produces the early exercise as well. Graph the early exercise curves produced.