CS 522 Assignment 2 March 1, 2001

The assignment should be done alone; however, discussions with fellow students are encouraged. Matlab programs should be sufficiently documented.

1. Consider an asset with zero volatility. We can explicitly calculate the future value of the asset and hence the value of a European call option on this asset. The value of the call option will then depend on the expected rate of return μ for this asset. On the other hand, we can use the explicit formula for the call option, in which μ does not appear. Explain the apparent contradiction.

2. Assume now that the underlying price satisfies the following stochastic equation

$$\frac{dS_t}{S_t} = (\mu - q)dt + \sigma(S_t, t)dX_t, \tag{1}$$

where the interest rate r and the continuous dividend yield q are positive constants, X_t is a standard Brownian motion, and $\sigma(S, t)$ is a deterministic function satisfying suitable conditions so that (1) admits a unique solution.

Suppose that European call/put of all exercise price E and expiry T are available. Let $V(E, T; S, t)$ denote the value of a European option on the asset S in which the strike E and the expiry T are considered as independent variables, and S, t are fixed asset price and time respectively (we are interested in computing $V(E, T; S_0, 0)$).

(a) Regarding S, t as fixed and E, T as independent variables, show that in the classical Black-Scholes setting ($q = 0$ and σ is a constant)

$$\frac{\partial V}{\partial T} - \frac{1}{2}\sigma^2 E^2 \frac{\partial^2 V}{\partial E^2} + rE\frac{\partial V}{\partial E} = 0.$$

(This is essentially Q19 in Chapter 5 of the textbook. Hint: using the Black-Scholes formula, show that $V(E, T; S, t) = Ef(\frac{S}{E}, T - t)$ for some function f. Use $V(E, T; S, t) = Ef(\frac{S}{E}, T - t)$ to relate the derivatives $\frac{\partial^2 V}{\partial E^2}$, $\frac{\partial V}{\partial S}$, $\frac{\partial V}{\partial t}$ to the partial derivatives $\frac{\partial^2 V}{\partial E^2}$, $\frac{\partial V}{\partial E}$, $\frac{\partial V}{\partial T}$, respectively and then substitute them into the Black-Scholes equation)

(b) Now let the asset price satisfies (1) with a deterministic volatility function $\sigma(S, t)$. Under some technical assumptions, it can be shown that the European option value $V(E, T; S, t)$ satisfies the following **forward** parabolic partial differential equation,

$$\frac{\partial V}{\partial T} - \frac{1}{2}\sigma(E, T)^2 E^2 \frac{\partial^2 V}{\partial E^2} + (r - q)E\frac{\partial V}{\partial E} + qV = 0. \tag{2}$$

(Note that the equation in (a) is a special case of (2). This equation first appeared in Dupire, Risk vol. 7, 1994; a derivation can also be found in Andersen and Brotherton-Ratcliffe, Journal of Computational finance, vol1, 1997.) Compared to the backward BS equation described in the lectures and the textbook, this forward equation is more effective in computing simultaneously European option premiums $V(E,T;S_0,0)$ for different strikes and maturities.

Justify the following boundary condition for a put option as

$$P(E,T;S_0,0) = Ee^{-rT} - S_0e^{-qT}, \quad \text{as } E \to +\infty$$

What are the appropriate initial condition and the boundary condition $P(E,T;S_0,0)$ at $E=0$? Explain your answer.

(c) Assume that $E_{\max}$ is sufficiently large and $T_{\max}$ is the maximum expiry of interest. Let $\{(E_i,T_j)\}_{i=0,\cdots,N}^{j=0,\cdots,M}$ denote a uniform discretization of the region $[0,E_{\max}]\times[0,T_{\max}]$, i.e., $\delta T = \frac{T_{\max}}{M}$, $\delta E = \frac{E_{\max}}{N}$. Let V_i^j denote the option value with the expiry $T_j = j\delta T$, and strike $E_i = i\delta E$ and σ_i^j denote the volatility $\sigma(E_i,T_j)$. Using the **Crank-Nicolson** finite difference, show that $x^j = [V_1^j;\cdots;V_{N-1}^j]$ satisfies the finite difference equation

$$A^j x^j = b^{j-1}, \quad j = 1,2,\cdots,M \tag{3}$$

where A^j is the tridiagonal matrix below

$$A^j = \text{diag}(e^j(2\!:\!N-1),-1) + \text{diag}(d^j,0) + \text{diag}(f^j(1\!:\!N-2),1)$$

with

$$e^j = [e_1^j;e_2^j;\cdots;e_{N-1}^j],\ d^j = [d_1^j;d_2^j;\cdots;d_{N-1}^j],\ f^j = [f_1^j;f_2^j;\cdots;f_{N-1}^j],$$

where, for $i = 1,2,\cdots,N-1$,

$$\begin{aligned} e_i^j &= -\tfrac{1}{4}(r-q)\tfrac{\delta T}{\delta E}E_i - \tfrac{1}{4}(\sigma_i^j)^2\tfrac{\delta T}{\delta E^2}E_i^2 \\ d_i^j &= 1 + \tfrac{1}{2}q\delta T + \tfrac{1}{2}(\sigma_i^j)^2\tfrac{\delta T}{\delta E^2}E_i^2 \\ f_i^j &= \tfrac{1}{4}(r-q)\tfrac{\delta T}{\delta E}E_i - \tfrac{1}{4}(\sigma_i^j)^2\tfrac{\delta T}{\delta E^2}E_i^2, \end{aligned}$$

and, for $i = 2,\cdots,N-2$,

$$b_i^{j-1} = V_i^{j-1} + \frac{1}{4}(\sigma_i^{j-1})^2E_i^2\delta T\left(\frac{V_{i-1}^{j-1} - 2V_i^{j-1} + V_{i+1}^{j-1}}{\delta E^2}\right) - \frac{1}{4}(r-q)E_i\delta T\frac{V_{i+1}^{j-1} - V_{i-1}^{j-1}}{\delta E} - \frac{1}{2}q\delta T V_i^{j-1},$$

and

$$\begin{aligned} b_1^{j-1} = \;& V_1^{j-1} + \tfrac{1}{4}(\sigma_1^{j-1})^2E_1^2\delta T\left(\tfrac{V_0^{j-1} - 2V_1^{j-1} + V_2^{j-1}}{\delta E^2}\right) - \tfrac{1}{4}(r-q)E_1\delta T\tfrac{V_2^{j-1} - V_0^{j-1}}{\delta E} - \tfrac{1}{2}q\delta T V_1^{j-1} \\ & - e_1^j V_0^j, \\ b_{N-1}^{j-1} = \;& V_{N-1}^{j-1} + \tfrac{1}{4}(\sigma_{N-1}^{j-1})^2E_{N-1}^2\delta T\left(\tfrac{V_{N-2}^{j-1} - 2V_{N-1}^{j-1} + V_N^{j-1}}{\delta E^2}\right) - \tfrac{1}{4}(r-q)E_{N-1}\delta T\tfrac{V_N^{j-1} - V_{N-2}^{j-1}}{\delta E} - \tfrac{1}{2}q\delta T V_{N-1}^{j-1} \\ & - f_{N-1}^j V_N^j. \end{aligned}$$

(d) Using your **Crank-Nicolson** finite difference method and the initial and boundary conditions for the forward equation (2), write a **Matlab** function

$$V = \textbf{FwdBSEuroPut}(S_0, \sigma, r, q, E_{\max}, T_{\max}, M, N)$$

which returns the $(N+1)$-by-$(M+1)$ European put option values matrix V; $V(i,j)$ is the put option value with strike $(i-1)\delta E$ and maturity $(j-1)\delta T$, $i = 1, \cdots, N+1$ and $j = 1, \cdots, M+1$. Here S_0 is the initial underlying asset price, σ is a $(N+1)$-by-$(M+1)$ matrix specifying the volatility value at each grid point, r and q are constant interest rate and continuous dividend yield, $E_{\max} > 0$ is sufficiently large, and $T_{\max}$ is the maximum expiry of interest. You are given Matlab functions **TriDiLU**, **LBidiSol**, and **UBidiSol** to solve (3).

To model the volatility smile, one may consider the following volatility function

$$\sigma(E,T) \equiv \sigma(E) = \frac{\alpha}{E}$$

Let $\alpha = 30$ and $\sigma(0)$ arbitrarily equal 100 in your implementation. In addition, set

$$S_0 = 100,\ E_{\max} = 2S_0,\ T_{\max} = 1,\ r = 0.0315,\ q = 0.02,\ M = 100,\ N = 200.$$

i. In **Matlab**, plot the function $\sigma(E)$ in the interval $[.9S_0, 1.1S_0]$;

ii. Display your computed put option value matrix V (you can use **mesh** command in **Matlab**);

iii. Tabulate the option values with $T = 1$ and strikes E_i which are in the interval $[.9S_0, 1.1S_0]$; the first column lists the strikes and the second column lists the corresponding option values.

iv. Now let $q = 0$ and all other parameters as before. Tabulate the put values with the strikes E_i in the interval $[.9S_0, 1.1S_0]$. Use put-call parity to compute the corresponding call values. Compute and plot the implied volatilities of these call options using the matlab function **BLSIMPV**.

v. Suppose that you want to compute the value of a put option at time $t = 0$ and some future time $0 < t < T$. Which partial differential equation would you use, the Black-Scholes backward equation or the forward equation (2)? Why?

Suggestions. To help you debug the program, you might want to consider some constant volatility examples and compare put option values computed from your program with the corresponding values from the function **blsprice** in the **Matlab** financial tool box. If you don't have access to the toolbox, you can easily implement the BS formula yourself.