§7. The axioms

For reference, we list here the axioms of ZFC and of some related theories; these are explained in much greater detail in Chapters I and III. After each axiom we list the section in Chapters I or III where it first occurs.

AXIOM 0. *Set Existence* (I §5).

$$\exists x\,(x = x).$$

AXIOM 1. *Extensionality* (I §5).

$$\forall x\,\forall y\,\big(\forall z\,(z \in x \leftrightarrow z \in y) \rightarrow x = y\big).$$

AXIOM 2. *Foundation* (III §4).

$$\forall x\,[\exists y\,(y \in x) \rightarrow \exists y\,(y \in x \,\wedge\, \neg\exists z\,(z \in x \,\wedge\, z \in y))].$$

AXIOM 3. *Comprehension Scheme* (I §5). For each formula ϕ with free variables among $x, z, w_1, \ldots, w_n$,

$$\forall z\,\forall w_1, \ldots, w_n\,\exists y\,\forall x\,(x \in y \leftrightarrow x \in z \,\wedge\, \phi).$$

AXIOM 4. *Pairing* (I §6).

$$\forall x\,\forall y\,\exists z\,(x \in z \,\wedge\, y \in z).$$

AXIOM 5. *Union* (I §6).

$$\forall \mathscr{F}\,\exists A\,\forall Y\,\forall x\,(x \in Y \,\wedge\, Y \in \mathscr{F} \rightarrow x \in A).$$

AXIOM 6. *Replacement Scheme* (I §6). For each formula ϕ with free variables among $x, y, A, w_1, \ldots, w_n$,

$$\forall A\,\forall w_1, \ldots, w_n\,[\forall x \in A\,\exists!y\,\phi \rightarrow \exists Y\,\forall x \in A\,\exists y \in Y\,\phi].$$

On the basis of Axioms 0, 1, 3, 4, 5 and 6, one may define $\subset$ (subset), 0 (empty set), S (ordinal successor; $S(x) = x \cup \{x\}$), and the notion of well-ordering. The following axioms are then defined.

AXIOM 7. *Infinity* (I §7).

$$\exists x\,\big(0 \in x \,\wedge\, \forall y \in x\,(S(y) \in x)\big).$$

AXIOM 8. *Power Set* (I §10).

$$\forall x\,\exists y\,\forall z\,(z \subset x \rightarrow z \in y).$$

AXIOM 9. *Choice* (I §6).

$$\forall A\, \exists R\, (R \text{ well-orders } A).$$

ZFC is the system of Axioms 0–9.

For technical reasons, it will sometimes be important to know that some of the results which we prove from ZFC do not in fact require all the axioms of ZFC; the reason for this is discussed at the end of I §4. We list here some abbreviations for commonly used subtheories of ZFC. ZF consists of Axioms 0–8, ZF $-$ P consists of Axioms 0–7, and ZFC $-$ P consists of Axioms 0–7 plus Axiom 9. By ZFC^-, ZF^-, ZF^- $-$ P, and ZFC^- $-$ P, we mean the respective theory (ZFC, ZF, ZF $-$ P, and ZFC $-$ P) with Axiom 2 (Foundation) deleted. Other abbreviations for weakenings of ZFC are usually self-explanatory. For example, ZF^- $-$ P $-$ Inf is ZF^- $-$ P with the Axiom of Infinity deleted.