§6. Relations, functions, and well-ordering

The following intuitive picture should emerge from §5. For a given $\phi(x)$, there need not necessarily exist a set $\{x: \phi(x)\}$; this collection (or class) may be too big to form a set. In some cases, for example with $\{x: x = x\}$, the collection is provably too big. Comprehension says that if the collection is a sub-collection of a given set, then it does exist. In certain other cases, e.g. where the collection is finite or is not too much bigger in cardinality than a given set, it should exist but the axioms of §5 are not strong enough to prove that it does. We begin this section with a few more axioms saying that certain sets which should exist do, and then sketch the development of some basic set-theoric notions using these axioms.

Axioms 4–8 of ZFC all say that certain collections do form sets. We actually state these axioms in the (apparently) weaker form that the desired collection is a subcollection of a set, since we may then apply Comprehension to prove that the desired set exists. Stating Axioms 4–8 in this way will make it fairly easy to verify them in the various interpretations considered in Chapters VI and VII.

AXIOM 4. *Pairing.*

$$\forall x\, \forall y\, \exists z\, (x \in z \wedge y \in z). \qquad \square$$

AXIOM 5. *Union.*

$$\forall \mathscr{F}\, \exists A\, \forall Y\, \forall x\, (x \in Y \wedge Y \in \mathscr{F} \rightarrow x \in A). \qquad \square$$

AXIOM 6. *Replacement Scheme.* For each formula ϕ without Y free, the universal closure of the following is an axiom:

$$\forall x \in A\, \exists ! y\, \phi(x, y) \rightarrow \exists Y\, \forall x \in A\, \exists y \in Y\, \phi(x, y). \qquad \square$$

By Pairing, for a given x and y we may let z be any set such that $x \in z \wedge y \in z$; then $\{v \in z: v = x \vee v = y\}$ is the (unique by Extensionality) set whose elements are precisely x and y; we call this set $\{x, y\}$. $\{x\} = \{x, x\}$ is the set whose unique element is x. $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$ is the *ordered pair* of x and y. One must check that

$$\forall x\, \forall y\, \forall x'\, \forall y'\, (\langle x, y\rangle = \langle x', y'\rangle \rightarrow x = x' \wedge y = y').$$

In the Union Axiom, we are thinking of $\mathscr{F}$ as a family of sets and postulate the existence of a set A such that each member Y of $\mathscr{F}$ is a subset of A. This justifies our defining the *union* of the family $\mathscr{F}$, or $\bigcup \mathscr{F}$, by

$$\bigcup \mathscr{F} = \{x: \exists Y \in \mathscr{F}\, (x \in Y)\};$$

this set exists since it is also

$$\{x \in A : \exists Y \in \mathscr{F}\,(x \in Y)\}.$$

When $\mathscr{F} \neq 0$, we let

$$\bigcap \mathscr{F} = \{x : \forall Y \in \mathscr{F}\,(x \in Y)\};$$

this set exists since, for any $B \in \mathscr{F}$, it is equal to

$$\{x \in B : \forall Y \in \mathscr{F}\,(x \in Y)\}$$

(so we do not appeal to the Union Axiom here). If $\mathscr{F} = 0$, then $\bigcup \mathscr{F} = 0$ and $\bigcap \mathscr{F}$ "should be" the set of all sets, which does not exist. Finally, we set $A \cap B = \bigcap \{A, B\}$, $A \cup B = \bigcup \{A, B\}$, and $A \setminus B = \{x \in A : x \notin B\}$.

The Replacement Scheme, like Comprehension, yields an infinite collection of axioms—one for each ϕ. The justification of Replacement is: assuming $\forall x \in A\, \exists ! y\, \phi(x, y)$, we can try to let $Y = \{y : \exists x \in A\, \phi(x, y)\}$; Y should be small enough to exist as a set, since its cardinality is $\leq$ that of the set A. Of course, by Replacement (and Comprehension),

$$\{y : \exists x \in A\, \phi(x, y)\}$$

does exist, since it is also $\{y \in Y : \exists x \in A\, \phi(x, y)\}$ for any Y such that $\forall x \in A\, \exists y \in Y\, \phi(x, y)$.

For any A and B, we define the *cartesian product*

$$A \times B = \{\langle x, y\rangle : x \in A \wedge y \in B\}.$$

To justify this definition, we must apply Replacement twice. First, for any $y \in B$, we have

$$\forall x \in A\, \exists ! z\,(z = \langle x, y\rangle),$$

so by Replacement (and Comprehension) we may define

$$\operatorname{prod}(A, y) = \{z : \exists x \in A\,(z = \langle x, y\rangle)\}.$$

Now,

$$\forall y \in B\, \exists ! z\,(z = \operatorname{prod}(A, y)),$$

so by Replacement we may define

$$\operatorname{prod}'(A, B) = \{\operatorname{prod}(A, y) : y \in B\}.$$

Finally, we define $A \times B = \bigcup \operatorname{prod}'(A, B)$.

We now review some other notions which may be developed on the basis of the Axioms 0, 1, 3, 4, 5, and 6. A *relation* is a set R all of whose elements are ordered pairs.

$$\operatorname{dom}(R) = \{x : \exists y(\langle x, y\rangle \in R)\}$$

and

$$\operatorname{ran}(R) = \{y : \exists x(\langle x, y\rangle \in R)\}.$$

These definitions make sense for any set R, but are usually used only when R is a relation, in which case $R \subset \operatorname{dom}(R) \times \operatorname{ran}(R)$. We define $R^{-1} = \{\langle x, y\rangle : \langle y, x\rangle \in R\}$, so $(R^{-1})^{-1} = R$ if R is a relation.

f is a *function* iff f is a relation and

$$\forall x \in \operatorname{dom}(f)\, \exists! y \in \operatorname{ran}(f) (\langle x, y\rangle \in f).$$

$f : A \to B$ means f is a function, $A = \operatorname{dom}(f)$, and $\operatorname{ran}(f) \subset B$. If $f : A \to B$ and $x \in R$, $f(x)$ is the unique y such that $\langle x, y\rangle \in f$; if $C \subset A$, $f \restriction C = f \cap C \times B$ is the *restriction* of f to C, and $f''C = \operatorname{ran}(f \restriction C) = \{f(x) \colon x \in C\}$. Many people use $f(C)$ for $f''C$, but the notation would cause confusion in this book since often elements of A will be subsets of A as well.

$f : A \to B$ is 1–1, or an *injection*, iff f^{-1} is a function, and f is onto, or a *surjection*, iff $\operatorname{ran}(f) = B$. $f : A \to B$ is a *bijection* iff f is both 1–1 and onto.

A *total ordering* (sometimes called a *strict* total ordering) is a pair $\langle A, R\rangle$ such that R totally orders A—that is, A is a set, R is a relation, R is transitive on A:

$$\forall x, y, z \in A\, (xRy \wedge yRz \to xRz),$$

trichotomy holds:

$$\forall x, y \in A\, (x = y \vee xRy \vee yRx),$$

and R is irreflexive:

$$\forall x \in A\, (\neg (xRx)).$$

As usual, we write xRy for $\langle x, y\rangle \in R$. Note that our definition does not assume $R \subset A \times A$, so if $\langle A, R\rangle$ is a total ordering so is $\langle B, R\rangle$ whenever $B \subset A$.

Whenever R and S are relations, and A, B are sets, we say $\langle A, R\rangle \cong \langle B, S\rangle$ iff there is a bijection $f\colon A \to B$ such that $\forall x, y \in A\, (xRy \leftrightarrow f(x)\, S f(y))$. f is called an *isomorphism* from $\langle A, R\rangle$ to $\langle B, S\rangle$.

We say R *well-orders* A, or $\langle A, R\rangle$ is a *well-ordering* iff $\langle A, R\rangle$ is a total ordering and every non-0 subset of A has an R-least element.

If $x \in A$, let $\operatorname{pred}(A, x, R) = \{y \in A : yRx\}$. This notation is used mainly when dealing with ordering. The basic rigidity properties of well-ordering are given as follows.

6.1. LEMMA. *If $\langle A, R\rangle$ is a well-ordering, then for all $x \in A$, $\langle A, R\rangle \not\cong \langle \operatorname{pred}(A, x, R), R\rangle$.*

PROOF. If $f\colon A \to \text{pred}(A, x, R)$ were an isomorphism, derive a contradiction by considering the R-least element of $\{y \in A\colon f(y) \neq y\}$. $\square$

6.2. LEMMA. *If $\langle A, R\rangle$ and $\langle B, S\rangle$ are isomorphic well-orderings, then the isomorphism between them is unique.*

PROOF. If f and g were different isomorphisms, derive a contradiction by considering the R-least $y \in A$ such that $f(y) \neq g(y)$. $\square$

The proofs of Lemmas 6.1 and 6.2 are examples of proofs by transfinite induction.

A basic fact about well-orderings is that any two are comparable:

6.3. THEOREM. *Let $\langle A, R\rangle$, $\langle B, S\rangle$ be two well-orderings. Then exactly one of the following holds*:

(a) $\langle A, R\rangle \cong \langle B, S\rangle$;
(b) $\exists y \in B(\langle A, R\rangle \cong \langle \text{pred}(B, y, S), S\rangle)$;
(c) $\exists x \in A(\langle \text{pred}(A, x, R), R\rangle \cong \langle B, S\rangle)$.

PROOF. Let

$$f = \{\langle v, w\rangle : v \in A \ \wedge w \in B \\ \wedge \langle \text{pred}(A, v, R), R\rangle \cong \langle \text{pred}(B, w, S), S\rangle\};$$

note that f is an isomorphism from some initial segment of A onto some initial segment of B, and that these initial segments cannot both be proper. $\square$

The notion of well-ordering gives us a convenient way of stating the Axiom of Choice (AC).

Axiom 9. *Choice.*

$$\forall A\, \exists R\, (R \text{ well-orders } A). \qquad \square$$

There are many equivalent versions of AC. See, e.g., [Jech 1973], [Rubin–Rubin 1963], or Exercises 9–11.

This book is concerned mainly with set theory with AC. However, it is of some interest that much of the elementary development of set theory does not need AC, so in this chapter we shall explicitly indicate which results have used AC in their proofs. AC is not provable in ZF; see [Jech 1973], or VII Exercise E3.