Chapter 2. Calculus and the Real Numbers

Section 1 establishes some conventions about sets and functions. The next three sections are devoted to constructing the real numbers as certain Cauchy sequences of rational numbers, and investigating their order and arithmetic. The rest of the chapter deals with the basic ideas of the calculus of one variable. Topics covered include continuity, the convergence of sequences and series of continuous functions, differentiation, integration, Taylor's theorem, and the basic properties of the exponential and trigonometric functions and their inverses. Most of the material is a routine constructivization of the corresponding part of classical mathematics; for this reason it affords a good introduction to the constructive approach.

We assume that the reader is familiar with the order and arithmetic of the integers and the rational numbers. For us, a *rational number* will be an expression of the form p/q, where p and q are integers with $q \neq 0$. Two rational numbers p/q and p'/q' are *equal* if $pq' = p'q$. The integer n is identified with the rational number $n/1$.

There are geometric magnitudes which are not represented by rational numbers, and which can only be described by a sequence of rational approximations. Certain such approximating sequences are called real numbers. In this chapter we construct the real numbers and study their basic properties. Then we develop the fundamental ideas of the calculus.

1. Sets and Functions

Before constructing the real numbers, we introduce some notions which are basic to much of mathematics.

The totality of all mathematical objects constructed in accordance with certain requirements is called a *set*. The requirements of the

E. Bishop et al., *Constructive Analysis*

construction, which vary with the set under consideration, determine the set. Thus the integers form a set, the rational numbers form a set, and (we anticipate here the formal definition of 'sequence') the collection of all sequences of integers is a set.

Each set will be endowed with a binary relation $=$ of *equality*. This relation is a matter of convention, except that it must be an *equivalence relation;* in other words, the following conditions must hold for all objects x, y, and z in the set:

(1.1) (i) $x=x$

(ii) If $x=y$, then $y=x$

(iii) If $x=y$ and $y=z$, then $x=z$.

The relation of equality given above for rational numbers is an equivalence relation. In this example there is a finite, mechanical procedure for deciding whether or not two given objects in the set are equal. Such a procedure will not exist in general: there are instances in which we are unable to decide whether or not two given elements of a set are equal; such an instance, in the theory of real numbers, will be given later.

We use the standard notation $a \in A$ to denote that a is an *element*, or *member*, of the set A, or that the construction defining a satisfies the requirements a construction must satisfy in order to define an object of A. We also use the notation $\{a_1, a_2, \ldots\}$ for a set whose elements can be written in a (possibly finite) list.

The dependence of one quantity on another is expressed by the basic notion of an operation. An *operation* from a set A into a set B is a finite routine f which assigns an element $f(a)$ of B to each given element a of A. This routine must afford an explicit, finite, mechanical reduction of the procedure for constructing $f(a)$ to the procedure for constructing a. If it is clear from the context what the sets A and B are, we sometimes denote f by $a \mapsto f(a)$, in order to bring out the form of $f(a)$ for a given element a of A. The set A is called the *domain* of the operation, and is denoted by $\mathrm{dmn} f$. In the most important case, we have $f(a)=f(a')$ whenever $a, a' \in A$ and $a=a'$; the operation f is then called a *function*, or a *mapping* of A into B, or a *map* of A into B. For two functions f, g from A into B, $f=g$ means that $f(a)=g(a)$ for each element a of A. Taken with this equality relation, the collection of all functions from A into B becomes a set.

The notation $f\colon A \to B$ indicates that f is a function from the set A to the set B.

A function x whose domain is the set $\mathbb{Z}^+$ of positive integers is called a *sequence*. The object $x_n \equiv x(n)$ is called the n^{th} *term* of the sequence. The finite routine x can be given explicitly, or it can be left to inference: for example, by writing the terms of the sequence in order

$$(x_1, x_2, \ldots)$$

until the rule of their formation becomes clear. Different notations for the sequence whose n^{th} term is x_n are: $n \mapsto x_n$, $(x_1, x_2, \ldots)$, $(x_n)_{n=1}^{\infty}$, and (x_n). Thus the sequence whose n^{th} term is n^2 can be written $n \mapsto n^2$, or $(1, 4, 9, \ldots)$, or $(n^2)_{n=1}^{\infty}$, or simply (n^2).

A *subsequence* of a sequence (x_n) consists of the sequence (x_n) and a sequence $(n_k)_{k=1}^{\infty}$ of positive integers such that $n_1 < n_2 < \ldots$. We identify such a subsequence with the sequence whose k^{th} term is x_{n_k}.

Sometimes we shall speak of sequences whose domain is some set of integers other than $\mathbb{Z}^+$. For example, we shall write $(x_n)_{n=0}^{\infty}$ to denote a mapping x from the set of nonnegative integers, where $x_n \equiv x(n)$ for each n.

Another example arises as follows. If n is a positive integer, then a *finite sequence* of *length* n is a function from the set $\{1, 2, \ldots, n\}$ into a set B.

The *cartesian product*, or simply the *product*, of sets $X_1, \ldots, X_n$ is defined to be the set

$$X \equiv X_1 \times X_2 \times \ldots \times X_n$$

of all ordered n-tuples $(x_1, \ldots, x_n)$ with $x_1 \in X_1$, $x_2 \in X_2, \ldots$, and $x_n \in X_n$. Elements $(x_1, \ldots, x_n)$ and $(x'_1, \ldots, x'_n)$ of the cartesian product are *equal* if the *coordinates* (or *components*) x_k and x'_k are equal elements of X_k for each k.

If x is a finite sequence of elements of a set B, then x can be identified with the element $(x(1), \ldots, x(n))$ of the cartesian product

$$B^n \equiv B \times B \times \ldots \times B,$$

where n is the length of x.

Returning to functions in general, we say that a function f: $A \to B$ maps A *onto* B if to each element b of B there corresponds an element a of A with $f(a) = b$. In other words, f maps A onto B if there is an operation g from B into A such that $f(g(b)) = b$ for each b in B. A set A is *countable* if there exists a mapping of $\mathbb{Z}^+$ onto A; intuitively, this means that the elements of A can be arranged in a sequence with possible duplications.

The elements of the cartesian product $\mathbb{Z} \times \mathbb{Z}$ of the set $\mathbb{Z}$ of integers with itself can be arranged in a sequence as follows. We order the elements (m, n) of $\mathbb{Z} \times \mathbb{Z}$, first according to the value of $|m| + |n|$, then according to the value of m, and finally according to the value of

n. This produces the sequence

(1.2) $((0,0), (-1,0), (0,-1), (0,1), (1,0), (-2,0), (-1,-1), \ldots)$,

in which each element of $\mathbb{Z} \times \mathbb{Z}$ occurs exactly once. In the sequence (1.2), omit every term (m,n) with $n=0$, and replace each term (m,n) with $n \neq 0$ by m/n; this produces the sequence

(1.3) $$(0/-1, 0/1, -1/-1, \ldots),$$

in which every expression p/q, with p and q integers and $q \neq 0$, occurs exactly once. Keeping only the term 0/1 of (1.3), and those terms for which $q>0$, $p \neq 0$, and p is relatively prime to q, we obtain a sequence

(1.4) $$(0/1, -1/1, 1/1, \ldots)$$

which has the property that for any given rational number r there exists exactly one term equal to r.

For each positive integer n, let $\mathbb{Z}_n$ be the set $\{0, 1, \ldots, n-1\}$. If there is a mapping of $\mathbb{Z}_n$ onto the set A, then we say that A has *at most n elements*. A set with at most n elements for some n is said to be *subfinite*, or *finitely enumerable*. Note that every subfinite or countable set has at least one element.

Before we introduce stronger notions than countability and subfiniteness, we must discuss the composition of functions. The *composition* of two functions $f\colon A \to B$ and $g\colon B \to C$ is the function $g \circ f\colon A \to C$ defined by

$$(g \circ f)(a) \equiv g(f(a)) \qquad (a \in A).$$

Composition is associative:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

whenever the compositions are defined.

If $f\colon A \to B$, $g\colon B \to A$, and $g(f(a)) = a$ for all a in A, then the function g is called a *left inverse* of f, and the function f is called a *right inverse* of g. (Note that f has a left inverse if and only if it is *one-one*, in the sense that $a = a'$ for all elements a, a' of A with $f(a) = f(a')$.) When g is both a left and a right inverse of f, then it is simply called an *inverse* of f; f is then called a *one-one correspondence*, or a *bijection*, and the sets A and B are said to be *in one-one correspondence* with each other.

A set which is in one-one correspondence with the set $\mathbb{Z}^+$ of positive integers is said to be *countably infinite*. For example, let f be the sequence (1.4), and define a function g from the set $\mathbb{Q}$ of rational numbers to $\mathbb{Z}^+$ by writing $g(r) \equiv n$, where n is the unique positive integer for which $f(n) = r$. Then g is an inverse of f; so that the set $\mathbb{Q}$

is countably infinite. A similar proof using (1.2) shows that $\mathbb{Z} \times \mathbb{Z}$ is countably infinite.

A set which is in one-one correspondence with $\mathbb{Z}_n$ is said to *have n elements*, and to be *finite*. Every finite set is countable.

It is not true that every countable set is either countably infinite or subfinite. For example, let A consist of all positive integers n such that both n and $n+2$ are prime; then A is countable, but we do not know if it is either countably infinite or subfinite. This does not rule out the possibility that at some time in the future A will have become countably infinite or subfinite; it is possible that tomorrow someone will show that A is subfinite. This set A has the property that if it is subfinite, then it is finite. Not all sets have this property.

2. The Real Number System

The following definition is basic to everything that follows.

(2.1) **Definition.** A sequence (x_n) of rational numbers is *regular* if

(2.1.1) $$|x_m - x_n| \leqq m^{-1} + n^{-1} \quad (m, n \in \mathbb{Z}^+).$$

A *real number* is a regular sequence of rational numbers. Two real numbers $x \equiv (x_n)$ and $y \equiv (y_n)$ are *equal* if

(2.1.2) $$|x_n - y_n| \leqq 2n^{-1} \quad (n \in \mathbb{Z}^+).$$

The set of real numbers is denoted by $\mathbb{R}$.

(2.2) **Proposition.** *Equality of real numbers is an equivalence relation.*

Proof: Parts (i) and (ii) of (1.1) are obvious. Part (iii) is a consequence of the following lemma.

(2.3) **Lemma.** *The real numbers $x \equiv (x_n)$ and $y \equiv (y_n)$ are equal if and only if for each positive integer j there exists a positive integer N_j such that*

(2.3.1) $$|x_n - y_n| \leqq j^{-1} \quad (n \geqq N_j).$$

Proof: If $x = y$, then (2.3.1) holds with $N_j \equiv 2j$.

Assume conversely that for each j in $\mathbb{Z}^+$ there exists N_j satisfying (2.3.1). Consider a positive integer n. If m and j are any positive integers with $m \geqq \max\{j, N_j\}$, then

$$\begin{aligned} |x_n - y_n| &\leqq |x_n - x_m| + |x_m - y_m| + |y_m - y_n| \\ &\leqq (n^{-1} + m^{-1}) + j^{-1} + (n^{-1} + m^{-1}) < 2n^{-1} + 3j^{-1}. \end{aligned}$$

Since this holds for all j in $\mathbb{Z}^+$, (2.1.2) is valid. □

Notice that the proof of Lemma (2.3) singles out a specific N_j satisfying (2.3.1). This situation is typical: every proof of a theorem which asserts the existence of an object must embody, at least implicitly, a finite routine for the construction of the object.

The rational number x_n is called the n^{th} *rational approximation* to the real number $x \equiv (x_n)$. Note that the operation from $\mathbb{R}$ to $\mathbb{Q}$ which takes the real number x into its n^{th} rational approximation is not a function.

For later use we wish to associate with each real number $x \equiv (x_n)$ an integer K_x such that

$$|x_n| < K_x \qquad (n \in \mathbb{Z}^+).$$

This is done by letting K_x be the least integer which is greater than $|x_1| + 2$. We call K_x the *canonical bound* for x.

The development of the arithmetic of the real numbers offers no surprises: we operate with real numbers by operating with their rational approximations.

(2.4) **Definition.** Let $x \equiv (x_n)$ and $y \equiv (y_n)$ be real numbers with respective canonical bounds K_x and K_y. Write

$$k \equiv \max\{K_x, K_y\}.$$

Let α be any rational number. We define

(a) $x + y \equiv (x_{2n} + y_{2n})_{n=1}^{\infty}$

(b) $xy \equiv (x_{2kn} y_{2kn})_{n=1}^{\infty}$

(c) $\max\{x, y\} \equiv (\max\{x_n, y_n\})_{n=1}^{\infty}$

(d) $-x \equiv (-x_n)_{n=1}^{\infty}$

(e) $\alpha^* \equiv (\alpha, \alpha, \alpha, \ldots)$.

(2.5) **Proposition.** *The sequences* $x+y$, xy, $\max\{x, y\}$, $-x$, *and* α^* *of Definition* (2.4) *are real numbers.*

Proof: (a) Write $z_n \equiv x_{2n} + y_{2n}$. Then $x + y \equiv (z_n)$. For all positive integers m and n,

$$\begin{aligned}|z_m - z_n| &\leqq |x_{2m} - x_{2n}| + |y_{2m} - y_{2n}| \\ &\leqq (2n)^{-1} + (2m)^{-1} + (2n)^{-1} + (2m)^{-1} = n^{-1} + m^{-1}.\end{aligned}$$

Thus $x + y$ is a real number.

(b) Write $z_n \equiv x_{2kn} y_{2kn}$. Then $xy \equiv (z_n)$. For all positive integers m and n,

$$\begin{aligned}|z_m - z_n| &= |x_{2km}(y_{2km} - y_{2kn}) + y_{2kn}(x_{2km} - x_{2kn})| \\ &\leqq k|y_{2km} - y_{2kn}| + k|x_{2km} - x_{2kn}| \\ &\leqq k((2km)^{-1} + (2kn)^{-1} + (2km)^{-1} + (2kn)^{-1}) = n^{-1} + m^{-1}.\end{aligned}$$

Thus xy is a real number.

(c) Write $z_n \equiv \max\{x_n, y_n\}$. Then $\max\{x, y\} \equiv (z_n)$. Consider positive integers m and n. For simplicity assume that

$$x_m = \max\{x_m, x_n, y_m, y_n\}.$$

Then

$$\begin{aligned}|z_m - z_n| &= |x_m - \max\{x_n, y_n\}| \\ &= x_m - \max\{x_n, y_n\} \leqq x_m - x_n \leqq n^{-1} + m^{-1}.\end{aligned}$$

Thus $\max\{x, y\}$ is a real number.

(d) For all positive integers m and n,

$$|-x_m - (-x_n)| = |x_m - x_n| \leqq m^{-1} + n^{-1}.$$

Thus $-x$ is a real number.

(e) This is obvious. □

There is no trouble in proving that $(x, y) \mapsto x + y$, $(x, y) \mapsto xy$, and $(x, y) \mapsto \max\{x, y\}$ are functions from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$; that $x \mapsto -x$ is a function from $\mathbb{R}$ to $\mathbb{R}$; and that $\alpha \mapsto \alpha^*$ is a function from $\mathbb{Q}$ to $\mathbb{R}$.

The operation

$$x \mapsto |x| \equiv \max\{x, -x\}$$

is therefore a function from $\mathbb{R}$ to $\mathbb{R}$, and the operation

$$(x, y) \mapsto \min\{x, y\} \equiv -\max\{-x, -y\}$$

is a function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$.

The next proposition states that the real numbers obey the same rules of arithmetic as the rational numbers.

(2.6) **Proposition.** *For arbitrary real numbers x, y, and z and rational numbers α and β,*

(a) $x + y = y + x$, $xy = yx$

(b) $(x+y)+z=x+(y+z), \quad x(yz)=(xy)z$

(c) $x(y+z)=xy+xz$

(d) $0^*+x=x, \quad 1^*x=x$

(e) $x-x=0^*$

(f) $|xy|=|x||y|$

(g) $(\alpha+\beta)^*=\alpha^*+\beta^*$, $(\alpha\beta)^*=\alpha^*\beta^*$, *and* $(-\alpha)^*=-\alpha^*$.

We omit the simple proofs of these results.

We shall use standard notations, such as $x+y+z$ and $\max\{x, y, z\}$, without further comment.

There are three basic relations defined on the set of real numbers. The first of these, the equality relation, has already been defined. The remaining relations, which pertain to order, are best introduced in terms of certain subsets $\mathbb{R}^+$ and $\mathbb{R}^{0+}$ of $\mathbb{R}$.

(2.7) **Definition.** A real number $x\equiv(x_n)$ is *positive*, or $x\in\mathbb{R}^+$, if

(2.7.1) $$x_n>n^{-1}$$

for some n in $\mathbb{Z}^+$. A real number $x\equiv(x_n)$ is *nonnegative*, or $x\in\mathbb{R}^{0+}$, if

(2.7.2) $$x_n\geqq -n^{-1} \quad (n\in\mathbb{Z}^+).$$

The following criteria are often useful.

(2.8) **Lemma.** *A real number $x\equiv(x_n)$ is positive if and only if there exists a positive integer N such that*

(2.8.1) $$x_m\geqq N^{-1} \quad (m\geqq N).$$

A real number $x\equiv(x_n)$ is nonnegative if and only if for each n in $\mathbb{Z}^+$ there exists N_n in $\mathbb{Z}^+$ such that

(2.8.2) $$x_m\geqq -n^{-1} \quad (m\geqq N_n).$$

Proof: Assume that $x\in\mathbb{R}^+$. Then $x_n>n^{-1}$ for some n in $\mathbb{Z}^+$. Choose N in $\mathbb{Z}^+$ with

$$2N^{-1}\leqq x_n-n^{-1}.$$

Then

$$x_m\geqq x_n-|x_m-x_n|\geqq x_n-m^{-1}-n^{-1}$$
$$\geqq x_n-n^{-1}-N^{-1}>N^{-1}$$

whenever $m\geqq N$. Therefore (2.8.1) is valid.

Conversely, if (2.8.1) is valid, then (2.7.1) holds with $n=N+1$. Therefore $x\in\mathbb{R}^+$.

Assume next that $x\in\mathbb{R}^{0+}$. Then for each positive integer n,

$$x_m \geqq -m^{-1} \geqq -n^{-1} \qquad (m \geqq n).$$

Therefore (2.8.2) is valid with $N_n \equiv n$.

Assume finally that (2.8.2) holds. Then if k, m, and n are positive integers with $m \geqq N_n$, we have

$$x_k \geqq x_m - |x_m - x_k| \geqq -n^{-1} - k^{-1} - m^{-1}.$$

Since m and n are arbitrary, this gives $x_k \geqq -k^{-1}$. Therefore $x\in\mathbb{R}^{0+}$. □

As a corollary of Lemma (2.8), we see that if x and y are equal real numbers, then x is positive if and only if y is positive, and x is nonnegative if and only if y is nonnegative.

It is not strictly correct to say that a real number (x_n) *is* an element of $\mathbb{R}^+$. An element of $\mathbb{R}^+$ consists of a real number (x_n) *and* a positive integer n such that $x_n > n^{-1}$, because an element of $\mathbb{R}^+$ is not presented until both (x_n) and n are given. One and the same real number (x_n) can be associated with two distinct (but equal) elements of $\mathbb{R}^+$. Nevertheless we shall continue to refer loosely to a positive real number (x_n). On those occasions when we need to refer to an n for which $x_n > n^{-1}$, we shall take the position that it was there implicitly all along.

The proof of the following proposition is now easy, and will be left to the reader. For convenience, $\mathbb{R}^*$ represents either $\mathbb{R}^+$ or $\mathbb{R}^{0+}$.

(2.9) **Proposition.** *Let x and y be real numbers. Then*

(a) $x+y\in\mathbb{R}^*$ *and* $xy\in\mathbb{R}^*$ *whenever* $x\in\mathbb{R}^*$ *and* $y\in\mathbb{R}^*$

(b) $x+y\in\mathbb{R}^+$ *whenever* $x\in\mathbb{R}^+$ *and* $y\in\mathbb{R}^{0+}$

(c) $|x|\in\mathbb{R}^{0+}$

(d) $\max\{x,y\}\in\mathbb{R}^*$ *whenever* $x\in\mathbb{R}^*$

(e) $\min\{x,y\}\in\mathbb{R}^*$ *whenever* $x\in\mathbb{R}^*$ *and* $y\in\mathbb{R}^*$.

We now define the order relations on $\mathbb{R}$.

(2.10) **Definition.** Let x and y be real numbers. We define

$$x>y \text{ (or } y<x) \qquad \text{if } x-y\in\mathbb{R}^+$$

and

$$x\geqq y \text{ (or } y\leqq x) \qquad \text{if } x-y\in\mathbb{R}^{0+}.$$

A real number x is *negative* if $x<0^*$ – that is, if $-x$ is positive.

Consider real numbers x, x', y, and y' such that (i) $x = x'$, $y = y'$, and $x > y$. We have

$$x' - y' = x - y \in \mathbb{R}^+$$

and therefore (ii) $x' > y'$. We express the fact that (ii) holds whenever (i) is valid by saying that $>$ is a *relation* on $\mathbb{R}$. More formally, a *relation* on a set X is a subset S of $X \times X$ such that if x, x', y, y' are elements of X with $x = x'$, $y = y'$, and $(x, y) \in S$, then $(x', y') \in S$.

We express the fact that $x > y$ if and only if $y < x$ by saying that $>$ and $<$ are *transposed relations.* Similarly, $\geqq$ and $\leqq$ are transposed relations.

If $x < y$ or $x = y$, then $x \leqq y$. The converse is not valid: as we shall see later, it is possible that we have $x \leqq y$ without being able to prove that $x < y$ or $x = y$. For this reason it was necessary to define the relations $<$ and $\leqq$ independently of each other.

The following rules for manipulating inequalities are easily proved from Proposition (2.9). We omit the proofs.

(2.11) **Proposition.** *For all real numbers x, y, z, and t,*

(a) *$x < z$ whenever either $x < y$ and $y \leq z$ or $x \leq y$ and $y < z$*

(b) *$x \leqq z$ whenever $x \leqq y$ and $y \leqq z$*

(c) *$x + y \leqq z + t$ whenever $x \leqq z$ and $y \leqq t$*

(d) *$x + y < z + t$ whenever $x \leqq z$ and $y < t$*

(e) *$xy \leqq zy$ whenever $x \leqq z$ and $y \geqq 0^*$*

(f) *$xy < zy$ whenever $x < z$ and $y > 0^*$*

(g) *if $x < y$, then $-x > -y$*

(h) *if $x \leqq y$, then $-x \geqq -y$*

(i) $\max\{x, y\} \geqq x$

(j) $\min\{x, y\} \leqq x$

(k) *if $x \leqq y$ and $y \leqq x$, then $x = y$*

(l) $|x| \geqq 0^*$

(m) $|x + y| \leqq |x| + |y|$.

An important property of the relation $<$, of which we shall make no use, is the *antisymmetry* property, which states that at most one of the relations $x < y$ and $y < x$ is valid for given real numbers x and y. This negative statement has no place in the affirmative mathematics we are trying to develop, except as motivation. Its place is taken by the affirmative statement (k) of Proposition (2.11). As a general principle, negative statements are only for counterexamples and motivation; they are not to be used in subsequent work.

(2.12) **Definition.** For real numbers x and y we write $x \neq y$ if and only if $x<y$ or $x>y$.

Inequality $\neq$ is a relation because both $<$ and $>$ are relations. As motivation we have the negative statement that at most one of the relations $x \neq y$, $x=y$ can hold for given real numbers x and y. In other words, at most one of the relations $x<y$, $x>y$, $x=y$ can hold. This is clear from the definitions.

The following proposition defines the *inverse* x^{-1} of a real number $x \neq 0^*$, and derives the basic properties of the operation $x \mapsto x^{-1}$.

(2.13) **Proposition.** *Let x be a nonzero real number (so that $|x| \in \mathbb{R}^+$). There exists a positive integer N with $|x_m| \geqq N^{-1}$ for $m \geqq N$. Define*

$$y_n \equiv (x_{N^3})^{-1} \qquad (n<N)$$

and

$$y_n \equiv (x_{nN^2})^{-1} \qquad (n \geqq N).$$

Then

$$x^{-1} \equiv (y_n)_{n=1}^{\infty}$$

is a real number which is positive if x is positive, and negative if x is negative; also $xx^{-1}=1^*$.

If t is any real number for which $xt=1^*$, then $t=x^{-1}$. The operation $x \mapsto x^{-1}$ is a function. If $x \neq 0$ and $y \neq 0$, then $(xy)^{-1}=x^{-1}y^{-1}$. If $\alpha \neq 0$ is rational, then $(\alpha^*)^{-1}=(\alpha^{-1})^*$. If $x \neq 0$, then $(x^{-1})^{-1}=x$.

Proof: Our definitions guarantee that $|y_n| \leqq N$ for all n.

Consider positive integers m and n. Write

$$j \equiv \max\{m, N\}, \qquad k \equiv \max\{n, N\}.$$

Then

$$\begin{aligned} |y_m - y_n| &= |y_m||y_n||x_{jN^2} - x_{kN^2}| \\ &\leqq N^2((jN^2)^{-1} + (kN^2)^{-1}) = j^{-1} + k^{-1} \leqq m^{-1} + n^{-1}. \end{aligned}$$

Therefore x^{-1} is a real number.

Assume now that $x>0^*$. Then by (2.8), $x_n>0$ for all sufficiently large n. Hence $y_n > K_x^{-1}$ (where K_x is the canonical bound for x) for all sufficiently large n. It follows from (2.8) that $x^{-1}>0^*$. A similar proof shows that $x^{-1}<0^*$ whenever $x<0^*$.

Let k be the maximum of the canonical bounds for x and x^{-1}. Write $xx^{-1} \equiv (z_n)$. Then

$$z_n \equiv x_{2nk} y_{2nk} \equiv x_{2nk}(x_{2nN^2k})^{-1} \qquad (n \geqq N).$$

Therefore

$$\begin{aligned} |z_n - 1^*| &= |x_{2nN^2k}|^{-1}|x_{2nk} - x_{2nN^2k}| \\ &\leqq |y_{2nk}|((2nk)^{-1} + (2nN^2k)^{-1}) \leqq n^{-1} \end{aligned}$$

for $n \geqq N$. It follows that $xx^{-1}=1^*$.

If t is any real number with $xt=1^*$, then

$$x^{-1}=x^{-1}(xt)=(x^{-1}x)t=(xx^{-1})t=t.$$

If $x=x'$, then

$$x'x^{-1}=xx^{-1}=1^*.$$

Therefore $x^{-1}=(x')^{-1}$. It follows that $x\mapsto x^{-1}$ is a function.

If $x\neq 0$ and $y\neq 0$, then

$$(xy)x^{-1}y^{-1}=xx^{-1}yy^{-1}=1^*.$$

Therefore $x^{-1}y^{-1}=(xy)^{-1}$.

If $\alpha\neq 0$ is rational, then $\alpha^*\equiv(\alpha,\alpha,\ldots)$. Therefore

$$(\alpha^*)^{-1}=(\alpha^{-1},\alpha^{-1},\ldots)=(\alpha^{-1})^*.$$

For each x in $\mathbb{R}^+$, x^{-1} is in $\mathbb{R}^+$, and thus $(x^{-1})^{-1}$ exists. Since $x^{-1}x=xx^{-1}=1^*$, it follows that $(x^{-1})^{-1}=x$. Similarly $(x^{-1})^{-1}=x$ if x is negative. Therefore $(x^{-1})^{-1}=x$ whenever $x\neq 0$. □

Of course, we often write x/y instead of xy^{-1} when x and y are real numbers with $y\neq 0$.

As the previous propositions show, $(\alpha\beta)^*=\alpha^*\beta^*$, $(\alpha+\beta)^*=\alpha^*+\beta^*$, $(-\alpha)^*=-\alpha^*$, $(|\alpha|)^*=|\alpha^*|$, and $(\alpha^{-1})^*=(\alpha^*)^{-1}$ for all rational numbers α and β. Also $\alpha\,\triangle\,\beta$ if and only if $\alpha^*\,\triangle\,\beta^*$, where $\triangle$ stands for any of the relations $=$, $<$, $>$, and $\neq$. This situation is expressed by saying that the map $\alpha\mapsto\alpha^*$ is an *order isomorphism* from $\mathbb{Q}$ into $\mathbb{R}$. This justifies identifying $\mathbb{Q}$ with a subset of $\mathbb{R}$, as we previously identified $\mathbb{Z}$ with a subset of $\mathbb{Q}$. Henceforth we make no distinction between a rational number α and the corresponding real number α^*.

The next lemma shows that the n^{th} rational approximation x_n to a real number $x\equiv(x_n)$ actually approximates x to within n^{-1}.

(2.14) **Lemma.** *For each real number $x\equiv(x_n)$, we have*

$$|x-x_n|\leqq n^{-1}\qquad(n\in\mathbb{Z}^+).$$

Proof: By (2.4) and the definition of $|\ |$, the m^{th} rational approximation to $n^{-1}-|x-x_n|$ is

$$n^{-1}-|x_{4m}-x_n|\geqq n^{-1}-((4m)^{-1}+n^{-1})=-(4m)^{-1}>-m^{-1}.$$

By (2.7), we have $n^{-1}-|x-x_n|\in\mathbb{R}^{0+}$. Therefore $|x-x_n|\leqq n^{-1}$. □

(2.15) **Lemma.** *If $x\equiv(x_n)$ and $y\equiv(y_n)$ are real numbers with $x<y$, then there exists a rational number α with $x<\alpha<y$.*

Proof: By (2.4), we have $y-x\equiv(y_{2n}-x_{2n})_{n=1}^{\infty}$. Since $y-x\in\mathbb{R}^{+}$, by (2.7) there exists n in $\mathbb{Z}^{+}$ with $y_{2n}-x_{2n}>n^{-1}$. Write

$$\alpha\equiv\tfrac{1}{2}(x_{2n}+y_{2n}).$$

Then

$$\alpha-x\geqq\alpha-x_{2n}-|x_{2n}-x|\geqq\tfrac{1}{2}(y_{2n}-x_{2n})-(2n)^{-1}>0.$$

Also,

$$y-\alpha\geqq y_{2n}-\alpha-|y_{2n}-y|\geqq\tfrac{1}{2}(y_{2n}-x_{2n})-(2n)^{-1}>0.$$

Therefore $x<\alpha<y$. □

As a corollary, for each x in $\mathbb{R}$ and r in $\mathbb{R}^{+}$ there exists α in $\mathbb{Q}$ with $|x-\alpha|<r$. Here is another corollary.

(2.16) **Proposition.** *If $x_1,\ldots,x_n$ are real numbers with $x_1+\ldots+x_n>0$, then $x_i>0$ for some i $(1\leqq i\leqq n)$.*

Proof: By (2.15), there exists a rational number α with $0<\alpha<x_1+\ldots+x_n$. For $1\leqq i\leqq n$ let a_i be a rational number with

$$|x_i-a_i|<(2n)^{-1}\alpha.$$

Then

$$\sum_{i=1}^{n}a_i\geqq\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}|x_i-a_i|>\tfrac{1}{2}\alpha.$$

Therefore $a_i>(2n)^{-1}\alpha$ for some i. For this i it follows that

$$x_i\geqq a_i-|x_i-a_i|>0.$$ □

(2.17) **Corollary.** *If x, y, and z are real numbers with $y<z$, then either $x<z$ or $x>y$.*

Proof: Since $z-x+x-y=z-y>0$, either $z-x>0$ or $x-y>0$, by (2.16). □

The next lemma gives an extremely useful method for proving inequalities of the form $x\leqq y$.

(2.18) **Lemma.** *Let x and y be real numbers such that the assumption $x>y$ implies that $0=1$. Then $x\leqq y$.*

Proof: Without loss of generality, we take $y=0$. For each n in $\mathbb{Z}^{+}$, either $x_n\leqq n^{-1}$ or $x_n>n^{-1}$. The case $x_n>n^{-1}$ is ruled out, since it implies that $x>0$. Therefore $-x_n\geqq-n^{-1}$ for all n, and so $-x\geqq0$. Thus $x\leqq0$. □

(2.19) **Theorem.** *Let (a_n) be a sequence of real numbers, and let x_0 and y_0 be real numbers with $x_0 < y_0$. Then there exists a real number x such that $x_0 \leqq x \leqq y_0$ and $x \neq a_n$ for all n in $\mathbb{Z}^+$.*

Proof: We construct by induction sequences (x_n) and (y_n) of rational numbers such that

(i) $x_0 \leqq x_n \leqq x_m < y_m \leqq y_n \leqq y_0 \quad (m \geqq n \geqq 1)$

(ii) $x_n > a_n$ or $y_n < a_n \quad (n \geqq 1)$

(iii) $y_n - x_n < n^{-1} \quad (n \geqq 1)$.

Assume that $n \geqq 1$ and that $x_0, \ldots, x_{n-1}$, $y_0, \ldots, y_{n-1}$ have been constructed. Either $a_n > x_{n-1}$ or $a_n < y_{n-1}$. In case $a_n > x_{n-1}$, let x_n be any rational number with $x_{n-1} < x_n < \min\{a_n, y_{n-1}\}$, and let y_n be any rational number with $x_n < y_n < \min\{a_n, y_{n-1}, x_n + n^{-1}\}$. Then the relevant inequalities are satisfied. In case $a_n < y_{n-1}$, let y_n be any rational number with $\max\{a_n, x_{n-1}\} < y_n < y_{n-1}$, and x_n any rational number with $\max\{a_n, x_{n-1}, y_n - n^{-1}\} < x_n < y_n$. Again, the relevant inequalities are satisfied. This completes the induction.

From (i) and (iii) it follows that

$$|x_m - x_n| = x_m - x_n < y_n - x_n < n^{-1} \qquad (m \geqq n).$$

Similarly $|y_m - y_n| < n^{-1}$ for $m \geqq n$. Therefore $x \equiv (x_n)$ and $y \equiv (y_n)$ are real numbers. By (i) and (iii), they are equal. By (i), $x_n \leqq x$ and $y_n \geqq y$ for all n. If $a_n < x_n$, then $a_n < x$ and so $a_n \neq x$. If $a_n > y_n$, then $a_n > y = x$ and so $a_n \neq x$. Thus x has the required properties. □

Theorem (2.19) is the famous theorem of Cantor, that the real numbers are uncountable. The proof is essentially Cantor's "diagonal" proof. Both Cantor's theorem and his method of proof are of great importance.

The time has come to consider some counterexamples. Let (n_k) be a sequence of integers, each of which is either 0 or 1, for which we are unable to prove either that $n_k = 1$ for some k or that $n_k = 0$ for all k. This corresponds to what Brouwer calls "a fugitive property of the natural numbers". For example, such a sequence can be defined as follows. Let n_k be 0 if $u^t + v^t \neq w^t$ for all integers u, v, w, t with $0 < u$, v, $w \leqq k$ and $3 \leqq t \leqq 2 + k$. Otherwise let n_k be 1. Then we are unable to prove $n_k = 1$ for some k, because this would disprove Fermat's last theorem. We are unable to prove $n_k = 0$ for all k, because this would prove Fermat's last theorem.

Now define $x_k \equiv 0$ if $n_j = 0$ for all $j \leqq k$, and $x_k \equiv 2^{-m}$ otherwise, where m is the least positive integer such that $n_m = 1$. Then $x \equiv (x_k)$ is a nonnegative real number, but we are unable to prove that $x>0$ or $x=0$. Since nothing is true unless and until it has been proved, it is untrue that $x>0$ or $x=0$.

Of course, if Fermat's last theorem is proved tomorrow, we shall probably still be able to define a fugitive sequence (n_k) of integers. Thus it is unlikely that there will ever exist a constructive proof that for every real number $x \geqq 0$ either $x>0$ or $x=0$. We express this fact by saying that there exists a real number $x \geqq 0$ such that it is *not* true that $x>0$ or $x=0$.

In much the same way we can construct a real number x such that it is *not* true that $x \geqq 0$ or $x \leqq 0$.

3. Sequences and Series of Real Numbers

We develop methods for defining a real number in terms of approximations by other real numbers.

(3.1) **Definition.** A sequence (x_n) of real numbers *converges* to a real number x_0 if for each k in $\mathbb{Z}^+$ there exists N_k in $\mathbb{Z}^+$ with

(3.1.1) $$|x_n - x_0| \leqq k^{-1} \quad (n \geqq N_k).$$

The real number x_0 is then called a *limit* of the sequence (x_n). To express the fact that (x_n) converges to x_0 we write

$$\lim_{n \to \infty} x_n = x_0$$

or

$$x_n \to x_0 \quad \text{as } n \to \infty$$

or simply $x_n \to x_0$.

A sequence (x_n) of real numbers is said to *converge*, or be *convergent*, if there exists a limit x_0 of (x_n).

It is easily seen that if (x_n) converges to both x_0 and x'_0, then $x_0 = x'_0$.

A convergent sequence is *bounded:* there exists r in $\mathbb{R}^+$ such that $|x_n| \leqq r$ for all n.

A convergent sequence of real numbers is not determined until the limit x_0 and the sequence (N_k) are given, as well as the sequence (x_n) itself. Even when they are not mentioned explicitly, these quantities are implicitly present. Similar comments apply to many subsequent definitions, including the following.