CS4860-Lect20-2018: Constructive Hyperreals

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Abstract

We will continue our discussion of non-standard models of first-order logics. We focus on the nonstandard real numbers, the hyperreals, since these ideas have been used by Keisler to teach college calculus as we discussed in Lectures 18 and 19, and they play an interesting role in modern logic.

We will also discuss *Skolem's Paradox* that arises from a countable model of the first-order set theory, *Zermelo-Fraenkel Set Theory with Choice*, ZFC. In ZFC we can define the classical real numbers, and they are uncountable, yet there is a countable model of ZFC. How can this be, that is the "paradox" we will discuss. The Smullyan text book discusses these concepts on pages 61,64, and 65.

Key Words: model theory, nonstandard models, infinitesimals, hyperreals, Skolem Paradox.

1 Background

Nonstandard first-order models date back to Cantor as he was investigating set theory. One of his early results is the claim that there are the same number of points in the line segment ab where a and b are known to be separated as there are in the unit square with side ab. Another early result was Skolem's theorem that if a first order theory has a model, then it has an uncountable model.

The concept of infinitesimals dates back much further to the efforts of Leibniz and Newton to develop the calculus as we have previously discussed. Interestingly, these ideas are still used in physics to teach the calculus. There are recent books on the history of infinitesimals that attempt to capture the radical nature of these ideas.

2 Axioms for the Real Numbers, $\mathbb{R}$

$\mathbb{R}$ is the set of all reals, $\mathbb{P}$ the positive reals.

A1. $\forall x, y : \mathbb{R}. \ (x + y = y + x)$
A2. $\forall x, y, z : \mathbb{R}. \ (x + y) + z = x + (y + z)$
A3. $0 \in \mathbb{R} \ \& \ \forall x. : \mathbb{R}. \ (x + 0 = x)$
A4. $\forall x : \mathbb{R}. \ \exists y : \mathbb{R}. \ (x + y = 0)$
A5. $\forall x, y : \mathbb{R}. \ (x \cdot y = y \cdot x)$
A6. $\forall x, y, z : \mathbb{R}. \ (x \cdot y) \cdot z = x \cdot (y \cdot z)$
A7. $1 \in \mathbb{R}. \ \& \ 1 \neq 0 \ \& \ \forall x : \mathbb{R}. \ (x \cdot 1 = x)$
A8. $\forall x : \mathbb{R}. \ (x \neq 0 \Rightarrow \exists y : \mathbb{R}. \ (x \cdot y = 1)$
A9. $\forall x, y, z : \mathbb{R}. \ (x \cdot (y + z) = x \cdot y + x \cdot z)$

Axioms of Order (ordered field)

$\forall x, y : \mathbb{P}.(x + y) \in \mathbb{P} \;\; \& \;\; (x \cdot y) \in \mathbb{P} \;\; \& \;\; -x \notin \mathbb{P}$

$\forall x : \mathbb{R}.\ x = 0 \;\vee\; x \in \mathbb{P} \;\vee\; -x \in \mathbb{P}$

Completeness Axiom - let S be a set of reals.

$\forall S.\ (S \subseteq \mathbb{R} \Rightarrow \exists y : \mathbb{R}.\ y \textit{ is the least upper bound of } S)$

We denote y by $\sup(S)$.

Archimedes Principle

$\forall x : \mathbb{R}.\ \exists n : \mathbb{N}.\ (x < n)$

Challenge: Find another more compact axiomatization of $\mathbb{R}$.

3 Axioms for infinitesimals and hyperreal numbers

Here is a review of the axioms that we have briefly discussed in the first lectures on non-standard real numbers.

$$\begin{array}{lll} 0 < x < \frac{1}{n} & n > 0 & x \text{ is a positive infinitesimal} \\ \frac{-1}{n} < x < 0 & n > 0 & x \text{ is a negative infinitesimal} \end{array}$$

I. Axioms for Infinitesimals

(a) There is a positive infinitesimal.

(b) If ϵ is a positive infinitesimal then $-\epsilon$ is a negative infinitesmial, and $r + \epsilon$ is hyperreal but not real.

(c) If $\epsilon > 0$ and $a > 0$, then $a \cdot \epsilon$ is a positive infinitesimal.

(d) If $\epsilon > 0$ then $\frac{1}{\epsilon}$ is a positive infinitesimal and $\frac{-1}{\epsilon}$ is a negative infinitesimal.

II. Algebraic Axioms for Hyperreals, $\mathbb{R}^*$

(a) $\forall x : \mathbb{R}.\ x \in \mathbb{R}^*$

(b) $\forall x, y : \mathbb{R}^*.\ x + y,\ x \cdot y,\ x - y$ belong to $\mathbb{R}^*$

(c) $\forall x : \mathbb{R}^*.\ x \neq 0 \Rightarrow \frac{1}{x} \in \mathbb{R}^*$

(d) The associative, commutative, distributive, and identity laws for $\mathbb{R}$ hold for $\mathbb{R}^*$.

III. Order Axioms for $\mathbb{R}^*$

1.	$a < b \ \& \ b < c \Rightarrow a < c$	(transitivity)
2.	$a < b \ \vee \ a = b \ \vee \ b < a$	(trichotomy)
3.	$a < b \Rightarrow a + c < b + c$	(sum)
4.	$a < b \ \& \ c > 0 \Rightarrow ac < bc$	(product law)
5.	$\forall x : \mathbb{R}^*. \ \forall n : \mathbb{N}^+. \ \exists b : \mathbb{R}^*. \ b > 0 \ \& \ b^n = a$	(root axiom)

Definition An element of $\mathbb{R}$ is a *positive infinitesimal iff* is it less than every positive real and greater than 0. It is *negative iff* it is greater than every negative real. The real number 0 is also considered to be an infinitesimal, the *zero infinitesimal*.

Two elements of $\mathbb{R}^*$ are infinitely close, $a \approx b$, *iff* their difference, $a - b$, is infinitesimal. We write $a \not\approx b$ when they are not infinitely close.

Standard Part Axiom: Every hyperreal number is infinitely close to exactly one real number.

Standard Part Operator: We denote the *standard part* of a hyperreal a by st(a).

Facts:

1) st(a) $\in \mathbb{R}$, $b = \text{st}(b) + \epsilon$ for some infinitesimal ϵ if b is a finite hyperreal number.

2) If $a \in \mathbb{R}$, then st(a)=a.

The Function Axiom $\forall f : \mathbb{R} \rightarrow \mathbb{R}. \ \exists f^* : \mathbb{R}^* \rightarrow \mathbb{R}^*$ called the *natural extension* of f. We discuss this extension when we examine examples. The idea is that for $x : \mathbb{R}$, $f(x) = f^*(x)$.

Solution Axiom The *hyperreal graph* of $y = f(x)$ is the set of hyperreal numbers (x_0, y_0) such that $y_0 = f^*(x_0)$.

4 A Constructive Account of Infinitesimals and Hyperreal Numbers

Dr. Mark Bickford has implemented in type theory the basic concepts needed to give a constructive account of the hyperreal numbers. We will examine these definitions and results to illustrate on-going research at the intersection of mathematics, logic, and computer science. We use the type $\mathbb{P}_1$ as the type of basic propositions. We will discuss these types and type constructors in more detail when we study type theory in the remaining lectures.

1. $\mathbb{R}^\star \ == \ \mathbb{N} \ \rightarrow \ \mathbb{R}$.
2. $\mathbb{R} \ \epsilon \ Type$.
3. $x(n) \ == \ ap(x; n)$.
4. $\forall x : \mathbb{R}^\star . \forall n : \mathbb{N}. ap(x; n) \ \epsilon \ \mathbb{R}$.
5. $R^\star(x, y) \ == \ \exists n : \mathbb{N}. \forall m : N_n . R(x(n), y(n))$.
6. $\forall R : (\mathbb{R} \ \rightarrow \ \mathbb{R} \ \rightarrow \ \mathbb{P}). \forall x, y : \mathbb{R}^\star . R^\star(x)(y) \ \epsilon \ \mathbb{P}$.
7. $(x = y) \ \Leftrightarrow \ \exists n : \mathbb{N}. \forall m. (m > n) \Rightarrow x(m) = y(m)$.
8. $\forall x, y : \mathbb{R}^\star \ \epsilon \ \mathbb{P}$.
9. $\forall x, y : \mathbb{R}^\star . (x = y) \Rightarrow (y = x)$.
10. Define $f^\star(x)$ as $\lambda(n. f(x(n)))$.
11. $is - standard(x) \ == \ \exists r : \mathbb{R}. x = (r)^\star$.

References