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Abstract

We will discuss the classical theory of nonstandard models of first-order theories. These models provide a foundation for *nonstandard analysis*. H.J. Keisler's book on calculus uses a rigorous notion of *infinitesimal* real numbers, called *hyperreal numbers*, to explain the calculus the way it was developed by Leibniz and Newton.

We will also briefly discuss how to include the notion of a *constructive infinitesimal* which will allow us to give a simple definition of continuity for computable functions on the constructive reals.

Key Words: constructive reals, model theory, nonstandard models, infinitesimals.

1 Topics

(1) Challenge - Can there be a countable model of the real numbers? Explain.

(2) Axioms for the real numbers $\mathbb{R}$.

(3) Axioms for the hyperreal numbers $\mathbb{R}^*$.

(4) Defining infinitesimals.

We can trace interest in nonstandard models back to results of Cantor as he was investigating set theory. One of his early results is the claim that there are the same number of points in the line segment ab where a and b are known to be separated as there are in the unit square with side ab. Another early result was Skolem's theorem that if a first order theory has a model, then it has an uncountable model.

The calculus was developed by Leibniz and Newton using the notion of "infinitely small numbers" called *infinitesimals* nowadays. In the rigorous development of the calculus, we typically use the "epsilon delta" arguments rather than infinitesimals. Here we repeat the comparison between a standard approach and the nonstandard approach to the notion of a *continuous function* on the interval $[a, b]$.

Definition 1: A function from reals to reals is *continuous* at x if and only if for every ϵ greater than 0, there is a real δ greater than 0, depending on ϵ such that if $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$. We say that $lim f(x) = f(c)$ *as x goes to c.*

Definition 2: A function from reals to reals is continuous at a point c if and only if whenever x is *infinitely close* to c, $f(x)$ is infinitely close to $f(c)$.

The second definition sounds very intuitive and is simpler, but what does the phrase "infinitely close" mean? If we have the idea of an infinitesimal around, as Leibniz did, then we know – the two values differ only by an infinitesimal.

Classical theorems in *model theory* provide a precise definition of the concept of an infinitesimal real number. This subject relies heavily on nonconstructive results about classical first-order logic. So the first order of business for us is to understand classical first-order logic using the constructive one. We will take up this topic first.

We also need to compare classical set theory and type theory. It is very interesting that Aczel has given a constructive account of set theory in type theory in a series of accessible articles [1, 2, 3, 4]. But we will not study this topic.

2 Axioms for the Real Numbers, $\mathbb{R}$

$\mathbb{R}$ is the set of all reals, $\mathbb{P}$ the positive reals.

A1. $\forall x, y : \mathbb{R}.\ (x + y = y + x)$
A2. $\forall x, y, z : \mathbb{R}.\ (x + y) + z = x + (y + z)$
A3. $0 \in \mathbb{R}\ \&\ \forall x. : \mathbb{R}.\ (x + 0 = x)$
A4. $\forall x : \mathbb{R}.\ \exists y : \mathbb{R}.\ (x + y = 0)$
A5. $\forall x, y : \mathbb{R}.\ (x \cdot y = y \cdot x)$
A6. $\forall x, y, z : \mathbb{R}.\ (x \cdot y) \cdot z = x \cdot (y \cdot z)$
A7. $1 \in \mathbb{R}.\ \&\ 1 \neq 0\ \&\ \forall x : \mathbb{R}.\ (x \cdot 1 = x)$
A8. $\forall x : \mathbb{R}.\ (x \neq 0 \Rightarrow \exists y : \mathbb{R}.\ (x \cdot y = 1)$
A9. $\forall x, y, z : \mathbb{R}.\ (x \cdot (y + z) = x \cdot y + x \cdot z)$

Axioms of Order (ordered field)

$\forall x, y : \mathbb{P}.(x + y) \in \mathbb{P} \quad \& \quad (x \cdot y) \in \mathbb{P} \quad \& \quad -x \notin \mathbb{P}$

$\forall x : \mathbb{R}.\ x = 0 \ \vee \ x \in \mathbb{P} \ \vee \ -x \in \mathbb{P}$

Completeness Axiom - let S be a set of reals.

$\forall S.\ (S \subseteq \mathbb{R} \Rightarrow \exists y : \mathbb{R}.\ y \textit{ is the least upper bound of } S)$

We denote y by $\sup(S)$.

Archimedes Principle

$\forall x : \mathbb{R}.\ \exists n : \mathbb{N}.\ (x < n)$

Challenge: Find another more compact axiomatization of $\mathbb{R}$.

3 Notes on Infinitesimals and Hyperreal Numbers, $\mathbb{R}^*$

Here is an account of Keisler's axiomatic account of non-standard analysis for his book *Foundations of Infinitessimal Calculus* [6] based on earlier work [5]. The preface of his book provides some historical background and very high praise for the logician Abraham Robinson. Here is a quotation from the preface:

"In 1960 Abraham Robinson (1918 - 1974) solved the three hundred year old problem of giving a rigorous development of the calculus based on infinitesimals. Robinson's achievement was one of the major mathematical advances of the twentieth century. This is an exposition of Robinson's infinitesimal calculus at the advanced graduate level. It is entirely self-contained but is keyed to the 2000 digital edition of my first year college text Elementary Calculus: An Infinitesimal Approach and the second printed edition. It is available free online at www.math.wisc.edu/ Keisler/calc." He goes on to say in this Preface:

"The basic concepts of the calculus were originally developed in the seventeenth and eighteenth centuries using the intuitive notion of an infinitesimal, culminating in the work of Gottfried Leibniz (1646 - 1716) and Isaac Newton (1643 - 1727).

3.1 Axioms for infinitesimals and hyperreal numbers

$0 < x < \frac{1}{n}$	$n > 0$	x is a positive infinitesimal
$\frac{-1}{n} < x < 0$	$n > 0$	x is a negative infinitesimal

I. Axioms for Infinitesimals

(a) There is a positive infinitesimal.

(b) If ϵ is a positive infinitesimal then $-\epsilon$ is a negative infinitesmial, and $r+\epsilon$ is hyperreal but not real.

(c) If $\epsilon > 0$ and $a > 0$, then $a \cdot \epsilon$ is a positive infinitesimal.

(d) If $\epsilon > 0$ then $\frac{1}{\epsilon}$ is a positive infinitesimal and $\frac{-1}{\epsilon}$ is a negative infinitesimal.

II. Algebraic Axioms for Hyperreals, $\mathbb{R}^*$

(a) $\forall x : \mathbb{R}.\ x \in \mathbb{R}^*$

(b) $\forall x, y : \mathbb{R}^*.\ x + y,\ \ x \cdot y,\ \ x - y$ belong to $\mathbb{R}^*$

(c) $\forall x : \mathbb{R}^*.\ x \neq 0 \Rightarrow \frac{1}{x} \in \mathbb{R}^*$

(d) The associative, commutative, distributive, and identity laws for $\mathbb{R}$ hold for $\mathbb{R}^*$.

III. Order Axioms for $\mathbb{R}^*$

1.	$a < b \ \& \ b < c \Rightarrow a < c$	(transitivity)
2.	$a < b \ \vee \ a = b \ \vee \ b < a$	(trichotomy)
3.	$a < b \Rightarrow a + c < b + c$	(sum)
4.	$a < b \ \& \ c > 0 \Rightarrow ac < bc$	(product law)
5.	$\forall x : \mathbb{R}^*. \ \forall n : \mathbb{N}^+. \ \exists b : \mathbb{R}^*. \ b > 0 \ \& \ b^n = a$	(root axiom)

Definition An element of $\mathbb{R}$ is a *positive infinitesimal iff* is it less than every positive real and greater than 0. It is *negative iff* it is greater than every negative real. The real number 0 is also considered to be an infinitesimal, the *zero infinitesimal*.

Two elements of $\mathbb{R}^*$ are infinitely close, $a \approx b$, *iff* their difference, $a - b$, is infinitesimal. We write $a \not\approx b$ when they are not infinitely close.

Standard Part Axiom: Every hyperreal number is infinitely close to exactly one real number.

Standard Part Operator: We denote the *standard part* of a hyperreal a by $\mathrm{st}(a)$.

Facts: 1) $\mathrm{st}(a) \in \mathbb{R}$, $b = \mathrm{st}(b) + \epsilon$ for some infinitesimal ϵ if b is a finite hyperreal number.

2) If $a \in \mathbb{R}$, then $\mathrm{st}(a) = a$.

The Function Axiom $\forall f : \mathbb{R} \to \mathbb{R}. \ \exists f^* : \mathbb{R}^* \to \mathbb{R}^*$ called the *natural extension* of f. We discuss this extension when we examine examples. The idea is that for $x : \mathbb{R}$, $f(x) = f^*(x)$.

Solution Axiom The *hyperreal graph* of $y = f(x)$ is the set of hyperreal numbers (x_0, y_0) such that $y_0 = f^*(x_0)$.

References

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