Applied Logic - CS4860-2018-Lecture 15

Robert L. Constable

Abstract

The main topic of the lecture is the syntactic embedding of classical first-order logic (FOL) into *intuitionistic first-order logic* (iFOL). This embedding induces an evidence based account of first-order logic grounded in computation. These insightful results are due to Kurt Gödel and the Russian logician Andrei N. Kolmogorov [5, 6]. They raise a natural question about the relevance of truth based semantics [7]. We will examine that issue about which the class might have interesting viewpoints that we should discuss.

1 Embedding FOL into iFOL

Here is Gödel's embedding as presented by Kleene on page 492 of his book [3, 4].

If P is an atomic formula, the P^o is P. If A and B are formulas, then:

- $(A \Rightarrow B)^o$ is $(A^o \Rightarrow B^o)$.
- $(A \& B)^o$ is $(A^o \;\&\; B^o)$.
- $(A \vee B)^o)$ is $\sim(\sim A^o \;\&\; \sim B^o)$.
- $(\sim A)^o$ is $\sim A^o$.
- $(\forall x.A(x))^o$ is $\forall x.A^o(x)$.
- $(\exists x.A(x))^o$ is $\sim\forall x.\sim A^o(x)$.

Kleene demonstrates in part (a) of his Theorem 60 page 495 that for any formula A of formal number theory, PA (for Peano Arithmetic), it is provable that A is equivalent to A^o.

In part (c) of his theorem he proves that if $H \vdash G$ classically, then $H^o \vdash G^o$ intuitionistically. He credits Gerhard Gentzen with this 1936 result [1].

There is another translation of A to an A' that is like the A^o version except that $A \Rightarrow B$ is translated as $\sim(A \;\&\; \sim B)$. For this translation we know that if $H \vdash G$ in PA, then $H' \vdash G'$ in HA [2].

References

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