Lecture 16

CS 4860

October 20, 2016

(1) Quick summary of FOL consistency and completeness.

What is the tricky part for completeness?

(2) Why is FOL so central in mathematical logic? It is because we can define important 'theories of mathematics" in it. For example, we will give the axioms for Peano Arithmetic. This is the theory of the natural number $\mathbb{N} = \{0, 1, 2, \dots\}$.

To give a theory of real numbers, $\mathbb{R}$, it is more natural to use Higher-Order Logic, HOL. On the other hand, the standard axiomatic account of Zermelo-Fraenkel Set Theory with the axiom of choice, ZFC, is done in FOL where the domain of discourse is the collection Set of all sets. It is commonly held that we can express "classical mathematics" in ZFC. These two key theories, PA and ZFC, are first-order theories. This is critical for ZFC.

One of the puzzling results of FOL is that every satisfiable formula has a model in a denumerable domain. In particular, the axioms of set theory, say just enough to do the reals, call it Z, has a countable model. This leads to a theory called *non-standard analysis*.

(3) An interesting project for those who like FOL completeness would be to compare the various textbook proofs. I could provide copies of at least three.

(4) The famous *incompleteness theorem* of Gödel, saying that PA is incomplete, applies smoothly to first-order theories, although Gödel had in mind the theory of *Principia Mathematica*. It also applies to the extremely simple theory called $\mathcal{Q}$ which we will present, alongside PA.

Let's write a succinct summary of FOL consistency and completeness for FOL along the lines we used for the Propositional Calculus tableau rules (PC).

- Consistency for PC

  If a formula A is provable, then we know that searching for FA will fail, so A cannot be falsified. Hence A is a tautology.

- Completeness for PC

  A proof by tableau rules attempts to find a falsifying truth assignment (in all possible ways). If the formula A is a tautology, this effort will fail, *so all branches of the tableau search must close*. This means that A is provable if it is a tautology.

For FOL consistency is also easy to show. If a tableau proof closes, the FA is unsatisfiable, hence A is valid. (See p.55 for Smullyan's proof.) In more detail, we have not missed a way to falsify. (Satisfiable FA $\Rightarrow$ open tableau for FA, so closed (provable) implies FA is unsatisfiable.)

- Completeness for FOL: If A is valid, then there is a proof.

  Thm3: If A is valid, then A is provable. Indeed, *the systematic tableau must terminate in a proof* p.60 (even the potentially ∞ *systematic* searches).

- Undecidable: There is no algorithm to decide whether a formula A is provable or not.

- Kleene's Versions : If an FOL formula A is irrefutable in FOL ($\sim F$ is unprovable) then A is satisfiable in $\mathbb{N}$. Every formula A which is valid in $\mathbb{N}$ is provable. p.389

- Corollary: If A is satisfiable in some non-empty domain, then it is satisfiable in $\mathbb{N}$. p.934

We will see how various theories are expressed in FOL.

(1) Peano Arithmetic - from Kleene

(2) Set Theory - ZFC

(3) Theory $\mathcal{Q}$ - an amazing small theory

(4) Real analysis -

Axioms for Theory Q (Boolos & Jeffrey p.158)

- Q1. $\forall x, y.(x' = y' \Rightarrow x = y)$
- Q2. $\forall x.(0 \neq x')$
- Q3. $\forall x.(x \neq 0 \Rightarrow \exists y.(x = y'))$
- Q4. $\forall x.(x + 0 = x)$
- Q5. $\forall x.(x + y' = (x + y)')$
- Q6. $\forall x.(x \cdot 0 = 0)$
- Q7. $\forall x, y.(x \cdot y' = (x \cdot y) + x)$

All recursive functions are representable in Q.
None of the following are theorems of $\not Q$

- $\forall x.(x \neq x')$
- $\forall x, y.(x + y = y + x)$
- $\forall x.(0 + x = x)$
- $\forall x.(x < x')$
- $\forall x, y.(x \cdot y = y \cdot x)$
- $\forall x.(0 \cdot x = 0)$

Theorem

(1) Q is not decidable

(2) Gödel : There is no consistent and complete axiomatizable extension of $\not Q$.

(3) All functions representable in Q are recursive.

We will study Q further in the coming weeks. It is a very interesting small theory.

Axioms for ZFC

Axiom 0.	$\exists x.(x = x)$	set existence
Axiom 1.	Extensionality	$\forall x, y.(\forall z(z \in x \iff z \in y) \Rightarrow x = y)$
Axiom 2.	Foundation	$\forall[\exists y.(y \in x) \Rightarrow \exists y.(y \in x \;\&\; \sim \exists z.(z \in x \;\&\; z \in y))]$
Axiom 3.	Comprehension	$\forall z.\forall w_1, \ldots, w_n.\exists y.\forall x.(x \in y \iff x \in z$ $\&\; \phi(x, z, w_1, \ldots, w_n))$
Axiom 4.	Pairing	$\forall x \forall y \exists z.(x \in z \;\&\; y \in z)$
Axiom 5.	Union	
Axiom 6.	Replacement	$\forall A.\forall w_1, \ldots, w_n.[\forall x \in A.\exists! y.\phi \Rightarrow \exists y.\forall x \in A.\exists y \in Y \phi]$
Axiom 7.	Infinity	$\exists x.(0 \in x \;\&\; \forall y \in x.(s(y) \in x))$ where $S(x) = X \cup \{X\}$
Axiom 8.	Power Set	$\forall x.\exists y.\forall z.(z \subset x \Rightarrow z \in y)$.
Axiom 9.	Choice	$\forall A.\exists R$ (R well-orders A).