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Krivine's intuitionistic proof of classical completeness (for countable languages)

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Abstract

In 1996, Krivine applied Friedman's *A-translation* in order to get an intuitionistic version of Gödel completeness result for first-order classical logic and (at most) countable languages and models. Such a result is known to be intuitionistically underivable (see J. Philos. Logic 25 (1996) 559), but Krivine was able to derive intuitionistically a weak form of it, namely, he proved that every consistent classical theory has a model. In this paper, we want to analyze the ideas Krivine's remarkable result relies on, ideas which where somehow hidden by the heavy formal machinery used in the original proof. We show that the ideas in Krivine's proof can be used to intuitionistically derive some (suitable variants of) crucial mathematical results, which were supposed to be purely classical up to now: the Ultrafilter Theorem for countable Boolean algebras, and the maximal ideal theorem for countable rings.
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1. Intuitionistic model theory

The first step in presenting Krivine's result is explaining what we mean by "intuitionistic proof of first-order classical completeness in the countable case". Thus, in this section, we outline an intuitionistic version of first-order model theory, following the ideas introduced in [1]. At the end of this section we will be able to state Krivine's completeness theorem.

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Let $\mathscr{L}$ be a first-order countable language over some (classically) complete subset of the set

$$\{\forall, \rightarrow, \bot, \neg, \&, \vee, \exists\}$$

of the first-order connectives. In the following, we will consider only the subset $\{\forall, \rightarrow, \bot\}$ which is clearly complete (however, see Section 1.2).

As in [4], we assume that $\mathscr{L}$ includes a fixed set of function symbols and a fixed set of predicate symbols which, for the sake of simplicity, are supposed to include at least the equality predicate.

As logical rules, we consider the intuitionistic first-order logical rules for equality, stating that the equality is an equivalence relation compatible with all function symbols and predicates, and the standard rules of introduction and elimination for each connective in $\{\forall, \rightarrow, \bot\}$. We include in the logical rules the *double-negation* axiom schema, saying that, for each formula A, $(\neg\neg A \rightarrow A)$ holds. As usual, for any formula B, by $\neg B$ we mean $(B \rightarrow \bot)$.

In the metatheory we use the same rules, except the *double-negation* axiom schema. That is, we reason intuitionistically.

We prefer speaking of the set $\mathscr{M}$ of closed formulas true in a model rather than of the model itself. In this way, we avoid dealing with many details about structures. Thus, we call "model" what in fact is the set of closed formulas satisfied by a standard model. We first introduce "model" axiomatically and then we explain how to turn a model in our sense into a model in the Tarski's sense of the word, and vice versa. Unless otherwise specified, from now on we will consider only closed formulas, only set of closed formulas, and only closed terms.

Definition 1.1 (Classical model). Let $\mathscr{L}$ be any first-order language, and $\mathscr{M}$ any set of closed formulas of $\mathscr{L}$.

(1) We say that $\mathscr{M}$ satisfies:
   - (a) ($\bot$-faithfulness) if $\bot \notin \mathscr{M}$;
   - (b) (double negation) if for all closed A, $(\neg\neg A \rightarrow A) \in \mathscr{M}$;
   - (c) ($\rightarrow$-faithfulness) if for all closed A and B, $(A \rightarrow B) \in \mathscr{M}$ if and only if $A \in \mathscr{M}$ yields $B \in \mathscr{M}$;
   - (d) ($\forall$-faithfulness) if for all A with only x free, $\forall x.A \in \mathscr{M}$ if and only if, for all closed terms t of $\mathscr{L}$, $A[x := t] \in \mathscr{M}$.
   - (e) (equality axioms) if for all closed instances E of an axiom for equality, $E \in \mathscr{M}$;

(2) A *classical model* for the language $\mathscr{L}$ is any set $\mathscr{M}$ of closed formulas, whose language $\mathscr{L}(\mathscr{M})$ is some extension of $\mathscr{L}$ with one or more fresh constants, and which satisfies all conditions above.

For any countable Tarski structure $\mathscr{U}$ (see [4]), we can define a classical model $\mathscr{M}$ by setting $\mathscr{L}(\mathscr{M}) = \mathscr{L}$ plus all names of elements of $\mathscr{U}$, and $\mathscr{M} = \mathsf{Th}(\mathscr{U})$ (the set of closed A in the language $\mathscr{L}(\mathscr{M})$ which are true in $\mathscr{U}$). Conversely, for any classical model $\mathscr{M}$, we can build a Tarski structure $\mathscr{U}$ such that $\mathscr{M} = \mathsf{Th}(\mathscr{U})$. The elements of

$\mathscr{U}$ are the closed terms of the language $\mathscr{L}(\mathscr{M})$ of $\mathscr{M}$, under the equivalence relation $a\equiv b\Leftrightarrow(a=b\in\mathscr{M})$. $\mathscr{U}$ is not empty because $\mathscr{L}(\mathscr{M})$ includes some fresh constant.

A notion that will be useful later, when discussing Krivine's completeness result, is that of *minimal* model, which is a slight variation of the notion of classical model.

Definition 1.2 (Minimal model). Let $\mathscr{L}$ be a first-order countable language. Then a *minimal* model is any set $\mathscr{M}$ of formulas of some extension of $\mathscr{L}$ with one or more fresh constant, which is required to enjoy all of the conditions for a classical model but $\bot$-faithfulness.

Thus, any classical model is a minimal model but a minimal model also satisfies the formula $\bot$. Let $\mathscr{L}'=\mathscr{L}$ plus some fresh constant. Since in the next validity Theorem 1.4, we will show that a minimal model is closed under the $\bot$-*elimination* rule, it is immediate to see that there is at least one (and, classically, exactly one) minimal model with language $\mathscr{L}'$ which is not a classical model, and hence is not a structure in Tarski standard sense. This is the *all-true model*, defined by setting $\mathscr{M}\equiv\{A\mid A$ closed formula of $\mathscr{L}'\}$. The structure $\mathscr{U}$ associated to the all-true minimal model $\mathscr{M}$ is a singleton, and the only n-ary predicate definable in $\mathscr{U}$ by some formula of $\mathscr{L}'$ is the all-true predicate. In any Tarski structure, instead, at least the all-true and the all-false predicates are definable.

We can introduce now the notion of truth for a (closed) formula and a sequent in a minimal and a classical model.

Definition 1.3 (Interpretation of a formula). Let A be a closed formula, $\Gamma\vdash A$ be a closed sequent, and $\mathscr{M}$ be any set of closed formulas. Then

(1) *A is true in $\mathscr{M}$* (notation $\mathscr{M}\Vdash A$) if and only if $A\in\mathscr{M}$;
(2) *Γ is true in $\mathscr{M}$* (notation $\mathscr{M}\Vdash\Gamma$) if and only if $\Gamma\subseteq\mathscr{M}$;
(3) *$\Gamma\vdash A$ is true in $\mathscr{M}$* if and only if *$\mathscr{M}\Vdash\Gamma$ yields $\mathscr{M}\Vdash A$*;
(4) $\Gamma\vdash A$ is *minimally* (*classically*) *valid* (notations $\Gamma\Vdash_{\mathrm{min}}A$ and $\Gamma\Vdash_{\mathrm{class}}A$) if and only it is true in all minimal (classical) models.

It is not difficult to check that any minimal model $\mathscr{M}$, and hence any classical one, is closed under all of the rules of classical logic, namely, if the sequent(s) in the premise(s) of a rule are true in $\mathscr{M}$ then also the sequent in the conclusion is.

Theorem 1.4 (Validity theorem). *Let $\Gamma\vdash A$ be any closed sequent. Then, if $\Gamma\vdash A$ is provable in classical logic, then $\Gamma\Vdash_{\mathrm{min}}A$ ($\Gamma\Vdash_{\mathrm{class}}A$).*

Proof. We argue by induction over the length of the proof of $\Gamma\vdash A$, that is, we prove that any minimal (classical) model $\mathscr{M}$ is closed under all classical rules. Let us show here just a few rules which are not completely trivial.

- ($\bot$-Elimination) By $\to$-*faithfulness* applied twice and the definition of negation, it is not difficult to see that $\bot\to\neg\neg A\equiv\bot\to((A\to\bot)\to\bot)\in\mathscr{M}$. By *double negation*, $(\neg\neg A\to A)\in\mathscr{M}$. Hence, we conclude $\bot\to A\in\mathscr{M}$, by using again $\to$-*faithfulness*.

- ($\rightarrow$-Introduction) Suppose that $\Gamma, A \vdash B$ is true in $\mathscr{M}$, that is, $\mathscr{M} \Vdash \Gamma, A$ yields $\mathscr{M} \Vdash B$. Then we want to prove that $\Gamma \vdash A \rightarrow B$ is true in $\mathscr{M}$. So, let us assume that $\mathscr{M} \Vdash \Gamma$ in order to prove that $\mathscr{M} \Vdash A \rightarrow B$, that is, $A \rightarrow B \in \mathscr{M}$. Then, by $\rightarrow$-*faithfulness*, it is sufficient to prove that if $A \in \mathscr{M}$ then $B \in \mathscr{M}$. So, let us assume that $A \in \mathscr{M}$. Then $\mathscr{M} \Vdash \Gamma$ and $\mathscr{M} \Vdash A$. Hence, the hypothesis yields that $\mathscr{M} \Vdash B$, that is, $B \in \mathscr{M}$.
- ($\forall$-Introduction) Suppose that $\Gamma \vdash A$ is derivable. According to the restriction to the $\forall$-introduction, Γ is closed, while A has at most x free. Assume that $\mathscr{M} \Vdash \Gamma$, that is, $\Gamma \subseteq \mathscr{M}$. Then, for every closed term t, the closed sequent $\Gamma \vdash A[x := t]$ is also derivable with a proof of the same length than the proof of $\Gamma \vdash A$. Then, by inductive hypothesis, $A[x := t] \in \mathscr{M}$. We conclude $\forall x.A \in \mathscr{M}$ by $\forall$-*faithfulness*, that is, $\mathscr{M} \Vdash \forall x.A$. □

Let $\mathscr{X}$ be any set of closed formulas, A any closed formula. From now on, we write $\mathscr{X} \vdash A$ to mean that there is a proof of A with assumptions in $\mathscr{X}$.

Definition 1.5 (Consistency). Let $\mathscr{X}$ and $\mathscr{X}'$ be any sets of closed formulas. Then

(1) $\mathscr{X}$ is *consistent* if $\mathscr{X} \nvdash \bot$, *inconsistent* if $\mathscr{X} \vdash \bot$;
(2) $\mathscr{X}$ and $\mathscr{X}'$ are *equiconsistent* if and only if $\mathscr{X} \vdash \bot \Leftrightarrow \mathscr{X}' \vdash \bot$.

We are now able to state the notion of completeness for a classical logic. Here, we propose three different definitions which are (only) classically equivalent.

Definition 1.6 (Classical completeness). Classical logic is *classically complete* if $\Gamma \Vdash_{\mathrm{class}} A$ yields $\Gamma \vdash A$, for all closed $\Gamma \vdash A$.

Definition 1.7 (Minimal completeness). Classical logic is *minimally complete* if $\Gamma \Vdash_{\mathrm{min}} A$ yields $\Gamma \vdash A$, for all closed $\Gamma \vdash A$.

Definition 1.8 (Maximal extension). Classical logic is enjoys the *maximal extension property* if any consistent set of closed formulas $\mathscr{X}$ is true in some classical model.

Classical completeness is known to be classically, yet not intuitionistically, derivable (for the first result see for instance [4] while, the second can be found in [3]). However, Krivine was able to prove intuitionistically maximal extension and minimal completeness, which are, classically, just minor variants of classical completeness.

Let us call a *closed classical theory* $\mathscr{T}$, or just theory for short, any set of closed formulas closed under derivation. That is, $\mathscr{T} \vdash A$ holds if and only if $A \in \mathscr{T}$, for all closed A. Any classical theory satisfies $\bot$-*elimination*, and hence it is inconsistent if and only if it contains all closed formulas.

There is a clear connection between a minimal model $\mathscr{M}$ in our sense and a theory. Indeed, the validity theorem above shows that any minimal model $\mathscr{M}$ is a theory.[1] On the other hand, it is not true that a theory $\mathscr{T}$ forms a minimal model since it is well

[1] Note that when a set of formulas $\mathscr{M}$ is a minimal model, then both $\mathscr{M} \vdash A$ and $\mathscr{M} \Vdash A$ mean $A \in \mathscr{M}$.

possible that it does not enjoy $\rightarrow$-*faithfulness* or $\forall$-*faithfulness*. In fact, we will prove minimal completeness by showing how to extend a theory $\mathcal{T}$ to a minimal model and maximal extension by showing how to extend a consistent theory $\mathcal{T}$ to a classical model.

1.1. The intuitionistic completeness proof

In this section we re-organize the proof of Krivine's completeness result, in order to stress the principle it relies on. First of all, we need to define formally some well-known concepts. So, let us sum up some trivialities about the constant $\bot$ and negation.

Definition 1.9 (Meta-DN and completeness). Let $\mathcal{T}$ be a classical theory and A be closed. Then

(1) (Meta-DN) $\mathcal{T}$ satisfies *metalinguistic double negation for A* if and only if $[(\mathcal{T}\vdash\neg A)\Rightarrow(\mathcal{T}\vdash\bot)]\Rightarrow(\mathcal{T}\vdash A)$. And $\mathcal{T}$ satisfies *meta*-DN if and only if $\mathcal{T}$ satisfies meta-DN for all closed A.
(2) (Completeness) $\mathcal{T}$ *is complete for A* or, equivalently, $\mathcal{T}$ *decides A*, if and only if $\mathcal{T}\vdash A$ or $\mathcal{T}\vdash\neg A$. And $\mathcal{T}$ is *complete* if and only if it is complete for all closed A.

Note that, if $\mathcal{T}\vdash A$, then $\mathcal{T}$ trivially satisfies meta-DN and completeness for A. Hence, any inconsistent classical theory satisfies trivially both meta-DN and completeness.

We can think of completeness as a sort of meta-linguistic excluded middle for a classical theory. Classically, meta-DN and completeness are equivalent, while intuitionistically it is only possible to prove that meta-DN is a consequence of completeness.

Lemma 1.10. *Any classical theory $\mathcal{T}$ satisfies meta*-DN *for a closed A if and only if it is complete for the same A*.

Proof (Classical). Assume that either $\mathcal{T}\vdash A$ or $\mathcal{T}\vdash\neg A$ holds, and suppose that $\mathcal{T}\vdash\neg A$ implies $\mathcal{T}\vdash\bot$ in order to prove that $\mathcal{T}\vdash A$. Now, if $\mathcal{T}\vdash A$ we are done. On the other hand if $\mathcal{T}\vdash\neg A$ then $\mathcal{T}\vdash\bot$ follows. In this case, we get $\mathcal{T}\vdash A$ by closure of $\mathcal{T}$ under derivation. This part of the proof is intuitionistic.

The converse is valid only classically. Suppose that meta-DN for $\mathcal{T}$ and A holds, and assume for contradiction that both $\mathcal{T}\nvdash A$ and $\mathcal{T}\nvdash\neg A$ hold. From the latter it follows $(\mathcal{T}\vdash\neg A)\Rightarrow(\mathcal{T}\vdash\bot)$ and so, by meta-DN for A, we obtain $\mathcal{T}\vdash A$, contradiction. □

In order to gain a better comprehension of the meaning of meta-DN let us consider the following equivalent formulation of meta-DN, which will be useful in Section 2.

Lemma 1.11 (Reformulation of meta-DN). *Let $\mathcal{T}$ be a theory and A be closed. Then meta*-DN *for A holds if and only if, whenever $\mathcal{T}\cup\{A\}$ is equiconsistent with $\mathcal{T}$, then $\mathcal{T}\vdash A$.*

Proof. Observe that $\mathscr{T}\cup\{A\}$ is equiconsistent with $\mathscr{T}$ if and only if $\mathscr{T}\cup\{A\}\vdash\bot$ implies $\mathscr{T}\vdash\bot$. And $\mathscr{T}\cup\{A\}\vdash\bot$ if and only if $\mathscr{T}\vdash\neg A$. Thus, $\mathscr{T}\cup\{A\}$ is equiconsistent with $\mathscr{T}$ if and only if $\mathscr{T}\vdash\neg A$ implies $\mathscr{T}\vdash\bot$. □

Thus, being $\mathscr{T}\cup\{A\}$ equiconsistent with $\mathscr{T}$ and the hypothesis of meta-DN are equivalent; since the conclusion is the same, the two principles are clearly equivalent.

The key property in (Henkin's version of) Gödel's classical completeness proof is the following.

Lemma 1.12 (Gödel's lemma). *Let $\mathscr{T}$ be a classical theory and A be closed. Then*

(1) *there is a classical theory $\mathscr{T}'$ extending $\mathscr{T}$, complete for A and equiconsistent with $\mathscr{T}$;*
(2) *being complete for A is a monotonic property, that is, if $\mathscr{T}$ is complete for A and $\mathscr{T}'$ extends $\mathscr{T}$, then $\mathscr{T}'$ is complete for A.*

Proof (Classical). (1) If $\mathscr{T}$ derives $\neg A$, we set $\mathscr{T}'\equiv\mathscr{T}$. Then $\mathscr{T}'$ decides A and it is equiconsistent with $\mathscr{T}$. If $\mathscr{T}$ does not derive $\neg A$, we define $\mathscr{T}'$ as the theory whose axioms are all the formulas in $\mathscr{T}\cup\{A\}$. Then $\mathscr{T}$ and $\mathscr{T}'$ are equiconsistent because both of them is consistent. Since $\mathscr{T}'\vdash A$, clearly $\mathscr{T}'$ decides A.

(2) Obvious. □

This lemma is not derivable intuitionistically. Indeed, we cannot decide in general whether $\mathscr{T}$ derives $\neg A$. Still, switching between completeness for A and the less informative property of meta-DN for A is sufficient in order to obtain a constructive version of Gödel's lemma (and, eventually, of minimal completeness).

Lemma 1.13 (meta-DN and monotonicity). *Let $\mathscr{T}$ be a classical theory and A a formula. Then*

(1) *there is a classical theory $\mathscr{T}'$ extending $\mathscr{T}$, equiconsistent with $\mathscr{T}$ and meta-*DN *for A;*
(2) *being meta-*DN *for A is a monotonic property for equiconsistent classical theories.*

Proof. (1) Call $H_A=[(\mathscr{T}\vdash\neg A)\Rightarrow(\mathscr{T}\vdash\bot)]$ the hypothesis of meta-DN for A. Let $\mathscr{T}_0$ be the set of closed formulas of $\mathscr{T}$, and $\mathscr{T}'$ be the classical theory whose axioms set $\mathscr{T}_1$ is defined by setting:

$$B\in\mathscr{T}_1 \text{ if and only if } (B\in\mathscr{T}_0) \text{ or } (B=A \text{ and } H_A \text{ holds}).$$

Note that we cannot, in general, effectively list the elements of $\mathscr{T}_1$, because we do not know whether H_A holds. All we do know is that if H_A holds, then $\mathscr{T}_1=\mathscr{T}_0\cup\{A\}$. We have to prove that $\mathscr{T}'$ is equiconsistent with $\mathscr{T}$ and that $\mathscr{T}'$ satisfies meta-DN for A.

Equiconsistency: Every finite subset of $\mathscr{T}_1$ is either included in $\mathscr{T}_0$, or it is included in $\mathscr{T}_0\cup\{A\}$, in which case H_A holds (proof by induction over the finite subset). Each proof in $\mathcal{T}'$ uses finitely many elements of $\mathcal{T}_1$ as assumptions. Therefore, either all assumptions of the proof are in $\mathcal{T}_0$, and the proof itself is in $\mathcal{T}$, or some assumption is A, and the proof is in $\mathcal{T}'$, and H_A holds. Assume now that we have a proof of $\bot$ in $\mathcal{T}'$. Then either we have a proof of $\bot$ in $\mathcal{T}$, and we are done, or we have a proof of $\bot$ in $\mathcal{T}'$, and H_A holds. By definition of $\mathcal{T}'$, from $\mathcal{T}' \vdash \bot$ we get $\mathcal{T} \vdash \neg A$. And from this latter and H_A we conclude again $\mathcal{T} \vdash \bot$.

Meta-DN *for A*: Assume $(\mathcal{T}' \vdash \neg A) \Rightarrow (\mathcal{T}' \vdash \bot)$. Then by equiconsistency of $\mathcal{T}$ and $\mathcal{T}'$ we obtain $(\mathcal{T}' \vdash \neg A) \Rightarrow (\mathcal{T} \vdash \bot)$. Hence $(\mathcal{T} \vdash \neg A) \Rightarrow (\mathcal{T} \vdash \bot)$ follows, since $\mathcal{T}$ is included in $\mathcal{T}'$ and so if $\mathcal{T} \vdash \neg A$ then $\mathcal{T}' \vdash \neg A$. Thus, H_A holds. We conclude that $\mathcal{T}_1 = \mathcal{T}_0 \cup \{A\}$, and that $\mathcal{T}'$ derives A.

(2) Suppose that meta-DN holds for $\mathcal{T}$ and A, that $\mathcal{T}$ is equiconsistent with $\mathcal{T}'$ and that $\mathcal{T} \subseteq \mathcal{T}'$. Now, assume that $\mathcal{T}' \vdash \neg A \Rightarrow \mathcal{T}' \vdash \bot$ holds in order to prove that $\mathcal{T}' \vdash A$. Then, $\mathcal{T} \vdash \neg A \Rightarrow \mathcal{T}' \vdash \bot$ follows because $\mathcal{T} \subseteq \mathcal{T}'$. Thus, we get $\mathcal{T} \vdash \neg A \Rightarrow \mathcal{T} \vdash \bot$ by equiconsistency between $\mathcal{T}$ and $\mathcal{T}'$. Eventually, we obtain $\mathcal{T} \vdash A$ by meta-DN for $\mathcal{T}$. Now $\mathcal{T}' \vdash A$ follows from $\mathcal{T} \subseteq \mathcal{T}'$. □

The rest of the proof works essentially as in Gödel–Henkin, but for a metalinguistic property stating that for classical theories, $\rightarrow$-*faithfulness* is a consequence of meta-DN.

Lemma 1.14 (Meta-DN and $\rightarrow$-faithfulness). *Let $\mathcal{T}$ be any classical theory that satisfies meta*-DN. *Then $\mathcal{T}$ is $\rightarrow$-faithful, and conversely.*

Proof. Assume that $\mathcal{T} \vdash A$ yields $\mathcal{T} \vdash B$ in order to prove $\mathcal{T} \vdash A \rightarrow B$. By meta-DN, it is enough to prove that $\mathcal{T} \vdash \neg(A \rightarrow B)$ yields $\mathcal{T} \vdash \bot$. Now, from $\mathcal{T} \vdash \neg(A \rightarrow B)$ and classical logic, we get $\mathcal{T} \vdash A$ and $\mathcal{T} \vdash \neg B$. So, from the assumption that $\mathcal{T} \vdash A$ yields $\mathcal{T} \vdash B$, we obtain $\mathcal{T} \vdash B$. Eventually, from $\mathcal{T} \vdash B$ and $\mathcal{T} \vdash \neg B$ we get $\mathcal{T} \vdash \bot$, as wished.

For the converse, assume $\rightarrow$-faithfulness, and that $\mathcal{T} \vdash \neg A$ implies $\mathcal{T} \vdash \bot$, in order to prove $\mathcal{T} \vdash A$. By $\rightarrow$-faithfulness we deduce $\mathcal{T} \vdash \neg\neg A$. Since $\mathcal{T}$ is a classical theory, we have $\mathcal{T} \vdash \neg\neg A \rightarrow A$. By closure of $\mathcal{T}$ under modus ponens, we conclude $\mathcal{T} \vdash A$. □

Lemma 1.15 ($\rightarrow$-faithfulness Lemma). *For any classical theory $\mathcal{T}$, there is a classical theory $\mathcal{U}$, extending $\mathcal{T}$, equiconsistent with $\mathcal{T}$ and $\rightarrow$-faithful.*

Proof. Let $A_0, A_1, A_2, \ldots$ be a list of all closed formulas of $\mathcal{L}$. Define a sequence $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots$ of theories, such that $\mathcal{T}_0 = \mathcal{T}$ and each theory $\mathcal{T}_{n+1}$ is constructed as in Lemma 1.13 like a theory including $\mathcal{T}_n$, equiconsistent with it, and meta-DN for A_n. Finally, set $\mathcal{U} = \bigcup_{n \in \omega} \mathcal{T}_n$. Then, $\mathcal{U}$ is a classical theory because it is union of a chain of classical theories. Moreover, all theories $\mathcal{T}_n$ are equiconsistent each other by construction and $\mathcal{U}$, being the union of a family of equiconsistent theories, is equiconsistent with each of them. Finally, for each A_n, $\mathcal{U}$ includes some equiconsistent theory meta-DN for A_n. By monotonicity and equiconsistency, $\mathcal{U}$ is meta-DN for all A_n. Hence, by the previous Lemma 1.14, for any formulas A and B, $\mathcal{U} \vdash A \rightarrow B$ if and only if $\mathcal{U} \vdash A$ yields $\mathcal{U} \vdash B$. □

Theorem 1.16 (Krivine completeness theorem). *Let $\mathscr{T}$ be a classical theory. There exists a minimal model $\mathscr{U}$ which both extends $\mathscr{T}$ and is equiconsistent with $\mathscr{T}$. If $\mathscr{T}$ is consistent, then $\mathscr{U}$ is a classical model.*

Proof (Intuitionistic). For each $\mathscr{T}$, we can constructively define a conservative (and hence equiconsistent) extension $\mathscr{H}$ of $\mathscr{T}$, having infinitely many new constants, and which is Henkin complete. By this we mean: $\mathscr{H}$ contains, for each closed universal formula in $\mathscr{L}(\mathscr{H})$, (the dual of) a Henkin axiom: $A[x:=c] \to \forall x \cdot A$, where c is the Henkin constant for $\forall x.A$. Such a (constructive) proof can be found in any textbook (for instance, see [4]). Then, by the previous Lemma 1.15, there is some $\to$-faithful extension $\mathscr{U}$ of $\mathscr{H}$ which is equiconsistent with $\mathscr{H}$, and hence with $\mathscr{T}$. Moreover, $\mathscr{U}$ is still Henkin complete since it includes all Henkin axioms in $\mathscr{H}$. By Henkin completeness, $\mathscr{U} \vdash \forall x.A$ if and only if $\mathscr{U} \vdash A[x:=t]$ for all (closed) terms t in $\mathscr{L}(\mathscr{U})$. Therefore $\mathscr{U}$ is $\to$-faithful by construction, and $\forall$-complete because it is a Henkin theory. Finally, $\mathscr{U}$ includes all equality and double-negation axioms because it is a classical theory. Moreover, if $\mathscr{T}$ is consistent then also $\mathscr{U}$ is. Thus, if $\mathscr{T}$ is consistent then $\mathscr{U}$ is a classical model. □

The previous Theorem 1.16 shows how to build a *classical* model for any consistent theory, that is, it shows that maximal extension holds. But we can strengthen this result and show how to build an actual proof of $\Gamma \vdash A$ when this sequent is true in all *minimal* models.

To obtain this result, let $\mathscr{T}$ be the classical theory with axiom set is $\Gamma \cup \neg\{A\}$. By the previous Theorem 1.16, we can extend $\mathscr{T}$ to a minimal model $\mathscr{M}$ including $\mathscr{T}$ and equiconsistent with $\mathscr{T}$. Since $\mathscr{M}$ is a minimal model, by the assumption that $\Gamma \vdash A$ is true in all minimal models and the fact that $\Gamma \subseteq \mathscr{T} \subseteq \mathscr{M}$ we obtain that $A \in \mathscr{M}$. So, the fact that $\neg A \in \mathscr{T} \subseteq \mathscr{M}$ yields $\bot \in \mathscr{M}$ by $\to$-faithfulness. Thus, we deduce $\mathscr{T} \vdash \bot$ by equiconsistency of $\mathscr{M}$ and $\mathscr{T}$. By definition of $\mathscr{T}$, we conclude $\Gamma, \neg A \vdash \bot$, and, eventually, $\Gamma \vdash A$ by classical logic. So we proved that also Minimal completeness holds.

It is useful to stress that the whole proof that $\Gamma \vdash A$ is provable whenever it is true in all minimal models is intutionistic. Hence, it defines, through realizability interpretation, an algorithm turning a model-theoretical proof of $\Gamma \Vdash_{\min} A$ into a classical proof of $\Gamma \vdash A$. Krivine proposed to call this algorithm a "decompiler", because it recovers a first order formal proof out of an informal proof which uses set theory; here we think of informal proofs as "compilations", in set theory, of formal proofs. It is a bit puzzling that, even if Krivine's algorithm was recently implemented by Raffalli, no explicit description of it is currently available.

1.2. About faithfulness for logical connectives

It is interesting to note that, when the theory $\mathscr{T}$ we start with is consistent, the classical model $\mathscr{U}$ that we build according to Theorem 1.16 is not only $\bot$-*faithful*, $\to$-*faithful* and $\forall$-*faithful* but also faithful with respect to all the other connectives but disjunction (for an introduction to the notion of faithfulness for a connective see [1]).

Indeed, if $\mathscr{T}$ is consistent, then also $\mathscr{U}$ is consistent and hence $\mathscr{U} \Vdash \bot$ is logically equivalent to falsum. Then, $\mathscr{U}$ is $\neg$-faithful, that is, $\mathscr{U} \Vdash \neg A$ if and only if $\mathscr{U} \nVdash A$. Indeed, $\neg A \equiv A \to \bot$; therefore $\mathscr{U} \Vdash \neg A$ if and only if $\mathscr{U} \Vdash A \to \bot$ if and only if $\mathscr{U} \Vdash A$ yields $\mathscr{U} \Vdash \bot$ if and only if $\mathscr{U} \Vdash A$ yields falsum if and only if $\mathscr{U} \nVdash A$.

Moreover, $\mathscr{U}$ is &-faithful, that is, $\mathscr{U} \Vdash A \& B$ if and only if $\mathscr{U} \Vdash A$ and $\mathscr{U} \Vdash B$. Indeed, $A \& B \equiv (A \to (B \to \bot)) \to \bot$ and, by using $\bot$-*faithfulness*, meta-DN for $\mathscr{U}$, and $\to$-*faithfulness*, it is not difficult to see that $\mathscr{U} \Vdash (A \to (B \to \bot)) \to \bot$ if and only if $\mathscr{U} \Vdash A$ and $\mathscr{U} \Vdash B$.

Finally, $\mathscr{U}$ is $\exists$-faithful, namely $\mathscr{U} \Vdash \exists x.A$ if and only if there exists a closed term t such that $\mathscr{U} \Vdash A[x := t]$, because it is a Henkin model. Indeed, $\exists x.A$ stays for $\neg \forall x. \neg A$. Moreover, for some Henkin constant c, both $\mathscr{U} \Vdash \neg A[x := c] \to \forall x. \neg A$ and $\mathscr{U} \Vdash \forall x. \neg A \to \neg A[x := c]$ are valid since the first is a Henkin axiom and the second follows by $\to, \forall$-*faithfulness* of $\mathscr{U}$. Hence, by $\to$-*faithfulness*, $\mathscr{U} \Vdash \forall x. \neg A$ if and only if $\mathscr{U} \Vdash \neg A[x := c]$. So, $\mathscr{U} \Vdash \exists x.A$ if and only if $\mathscr{U} \Vdash \neg \forall x. \neg A$ if and only if $\mathscr{U} \Vdash \forall x. \neg A$ yields $\mathscr{U} \Vdash \bot$ if and only if $\mathscr{U} \Vdash \forall x. \neg A$ yields false if and only if $\mathscr{U} \Vdash \neg A[x := c]$ yields false if and only if $\mathscr{U} \Vdash A[x := c]$ (by meta-DN for $\mathscr{U}$).

Our model, however, is not intuitionistically faithful for disjunction. Indeed, suppose that we can prove that for all A, from $\mathscr{U} \Vdash A \vee \neg A$ we get $\mathscr{U} \Vdash A$ or $\mathscr{U} \Vdash \neg A$. Using realizability interpretation of intuitionism, from such a proof we can define a recursive map deciding whether $\mathscr{U} \Vdash A$ or $\mathscr{U} \Vdash \neg A$, i.e., $\mathscr{U} \nVdash A$. Therefore, $\mathscr{U}$ would be a recursive consistent complete extension of the original consistent theory $\mathscr{T}$. When $\mathscr{T}$ is Peano arithmetic, this is in contradiction with Gödel incompleteness theorem.

2. The ultrafilter theorem

In this section we try to isolate the constructive principle making a constructive proof of completeness possible, with the hope of applying them to other results as well. The main principle we used seems to have been the fact every filter over a countable Boolean algebra can be extended to a maximal equiconsistent filter; in fact, we applied this result to a classical theory, which is a particular case of Boolean algebra. A main consequence of this fact is that any consistent filter of a countable Boolean algebra can be extended to an ultrafilter. This theorem is well-known, but it was considered purely classical up to now.

Definition 2.1. Let $\mathscr{B} \equiv (B, \wedge, 1_B, \vee, 0_B, \neg)$ be a Boolean algebra. Then

(1) a *filter* F over $\mathscr{B}$ is a non-empty subset of B closed upwards and under finite intersection;

(2) If $X \subseteq B$, then

$$\uparrow(X) \equiv \{z \in B \mid \exists y_1, \ldots, y_k \in X. y_1 \wedge \cdots \wedge y_k \leqslant z\}$$

is the smallest filter including X. If F is a filter and x is an element in B, then by (F, x) we mean $\uparrow(F \cup \{x\})$;

(3) a filter F is *consistent* if it does not include 0_B, *inconsistent* if it does;

(4) two filters F and G are *equiconsistent* when F is inconsistent if and only if G is inconsistent;
(5) a filter F is *complete for an element* $x \in B$ if and only if, whenever F and (F, x) are equiconsistent, $x \in F$.
(6) a filter F is *complete* if it is complete for any element $x \in B$;
(7) a filter F is an *ultrafilter* if and only if it a maximal consistent filter, that is, F is consistent and if G is a consistent filter which includes F then $G = F$.

All definitions above are taken from standard mathematics textbook, but for completeness for a filter F, which is usually stated as: for all x, either $x \in F$ or $\neg x \in F$. After Lemmas 1.10 and 1.11 it is easy to check that the definition of completeness that we chosen is classically equivalent to the standard one, even if it is intuitionistically weaker. The fact that it is weaker is essential in order to prove constructively the ultrafilter theorem for countable Boolean algebra; in fact, it cannot be proved with the original definition.

Theorem 2.2. *Let F be any filter over a countable Boolean algebra $\mathscr{B}$. Then F can be extended to a complete filter Z equiconsistent with F.*

Proof (Intuitionistic). Let $x_1, \ldots, x_n, \ldots$ be any enumeration of the elements of B and define a filter chain F_n, by setting

$$F_0 = F,$$
$$F_{n+1} = \uparrow (F_n \cup \{y \in B \mid (y = x_n) \wedge H_n\}),$$

where H_n is a shorthand for "F_n and (F_n, x_n) are equiconsistent", that is, the hypothesis of the condition of completeness of F_n for x_n.

Note that the set $\{y \in B \mid (y = x_n) \wedge H_n\}$ is included in $\{x_n\}$, but we cannot, in general, effectively list its elements, because we do not know whether H_n holds.

Finally set $Z \equiv \bigcup_{n \in \omega} F_n$. Our thesis is that Z is a complete filter equiconsistent with F.

Equiconsistency: We prove first by induction on n that all F_n are equiconsistent with F (and hence also one each other). For $n = 0$ this holds trivially since $F_0 = F$. Suppose now that F_n is equiconsistent with F. Assume that F_{n+1} is inconsistent, that is, $(y_1 \wedge \cdots \wedge y_k) \leqslant 0_B$ for some $y_1, \ldots, y_k$ generators of F_{n+1}. We have to prove that 0_B is in F_n, and hence in F. By induction we can prove that any finite intersection of generators of F_{n+1} is either in F_n, or it has the shape $y \wedge x_n$ for some $y \in F_n$, in which case H_n holds. In the first case, 0_B is in F_n and we are done. In the latter case H_n holds, and $y \wedge x_n \leqslant 0_B$ for some $y \in F_n$. Therefore we obtain $y \leqslant (x_n \to 0_B) = \neg x_n$ since $\mathscr{B}$ is a Boolean algebra. But F_n is a filter and thus $\neg x_n$ is in F_n. So, we deduce that (F_n, x_n) is inconsistent. Thus, by H_n we conclude again that also F_n is inconsistent. On the other hand, if F_n is inconsistent also F_{n+1} is inconsistent since $F_n \subseteq F_{n+1}$.

We can now conclude that Z, being a union of equiconsistent filters, is a filter equiconsistent with each of them.

Completeness: Take any $x \in B$, and assume that Z and (Z,x) are equiconsistent. We want to prove that $x \in Z$. Observe that $x = x_n$, for some $n \in \omega$. If we manage to prove that F_n and (F_n, x_n) are equiconsistent, then we will conclude that $x_n \in F_{n+1}$ and hence $x \in Z$ since $F_{n+1} \subseteq Z$. So, let us assume that (F_n, x_n) is inconsistent; then also (Z,x) is inconsistent since F_n is included in Z and hence $(Z,x) \supseteq (F_n, x_n)$. Therefore the assumption that (F_n, x_n) is inconsistent and the equiconsistency of Z and (Z,x) yield that Z is inconsistent. Hence, F_n is inconsistent by equiconsistency of Z and F_n. On the other hand, if we assume that F_n is inconsistent we immediately obtain that (F_n, x_n) is inconsistent since $F_n \subseteq (F_n, x_n)$. □

The following corollary is now immediate.

Corollary 2.3 (The countable ultrafilter theorem). *Let F be a consistent filter over a countable Boolean algebra. Then F can be extended to an ultrafilter.*

Proof. By the previous theorem, F can be extended to a consistent and complete filter Z. Suppose now that G is a consistent filter including Z. Then we have to prove that $G \subseteq Z$. So, let us assume that $x \in G$ in order to prove that $x \in Z$. Since Z is complete this amounts to prove that Z is equiconsistent with (Z,x), that is, to prove that (Z,x) is consistent since we already know that Z is consistent. But $Z \subseteq G$ and $x \in G$ and so (Z,x) is included in the consistent filter G and hence it is consistent. □

2.1. Some comments from a constructive viewpoint

Consider the case the filter F is consistent. The main problem, from a constructive viewpoint, in the definition of the ultrafilter Z in the Corollary 2.3 above is that, for all $x \in B$, $\neg x \in Z$ if and only if $(Z, \neg x)$ is consistent if and only if $x \notin Z$. This means that membership to Z is a non-informative predicate since it is equivalent to a negation. So, any hypothetical proof for an assumption of the form $a \in Z$ does not depend on a. Thus, we wonder if the ultrafilter theorem can be considered, from a constructive standpoint something more than a (nice) curiosity. Still, if one is trying to prove constructively that a filter F is inconsistent, to extend it to an ultrafilter can be the right way to obtain the result.

Notice that, on the other hand, the statement "F is inconsistent" means that $0_B \in F$. Thus "F is inconsistent" can be an informative statement, that is, it can be associated to a witness. In Krivine's proof of completeness, inconsistency of the filter associated to the classical theory with axioms $\Gamma \cup \neg\{A\}$ carried, in fact, a witness, in form of a syntactic object, that is, a proof of $\Gamma \cup \neg\{A\} \vdash \bot$, and, from it, it was possible to obtain a classical proof of $\mathcal{T} \vdash A$.

3. The maximal ideal theorem

Some results, which are in classical mathematics easy consequences of some equivalent formulation of the ultrafilter theorem for Boolean algebras, can be taken into the realm of constructive mathematics as they are. Some others require more detailed constructions. We devote this section to an example of the first kind: the maximal ideal theorem for countable rings.

3.1. The maximal ideal construction

Let us begin by recalling some standard definitions about ideals of a ring. An *ideal* I of a ring $\mathscr{A} \equiv (A, +, \cdot, -, 0_A, 1_A)$ is a subset closed under 0_A, opposite, sum, and such that $x \in I$ and $a \in A$ yield $ax \in I$. Let us call *inconsistent* any ideal I of $\mathscr{A}$ including 1_A, that is, I is inconsistent if and only if $I = A$. Let *consistent* mean "not inconsistent", that is, I is consistent if it does not include 1_A. Let *equiconsistent* for two ideals I and J means that $1_A \in I$ if and only if $1_A \in J$. Let us denote by (X) the minimal ideal including the subset X of A, and by (I, x) the minimal ideal including the ideal I and the element x, that is, $(I, x) = (I \cup \{x\})$.

Lemma 3.1 (Construction of (X)). *Let X be a subset of the ring $\mathscr{A}$. Then*

$$(X) = \{a_1 y_1 + \cdots + a_k y_k \mid y_1, \ldots, y_k \in X, a_1, \ldots, a_k \in A\}$$

is the minimal ideal containing X.

Let us call *complete for x* an ideal I such that if (I, x) is consistent, then $x \in I$ and *complete* an ideal which is complete for all $x \in A$. Note that an inconsistent ideal is trivially complete. Let us call *maximal* any consistent ideal which is maximal by inclusion.

Lemma 3.2 (Maximality). *An ideal M is maximal if and only if it is complete and consistent.*

Proof. From left to right. Let M be maximal. Then M is consistent by definition. In order to prove completeness, let x be an element of A. Then (M, x) includes M. Suppose now that (M, x) is consistent. Then, by maximality, $M = (M, x)$ and hence x belongs to M.

From right to left. Assume that M is consistent and complete and suppose that I is a consistent ideal including M. Then, we have to prove that $I = M$, namely, that $I \subseteq M$. Assume that $x \in I$ in order to prove that $x \in M$. To this aim, consider the ideal (M, x). Since $M \subseteq I$ and $x \in I$, (M, x) is included in I and therefore it is consistent. Thus, by completeness of M, (M, x) is included in M and hence $x \in M$. □

It is possible to prove also that completeness is monotone.

Lemma 3.3 (Monotonicity). *Let I and J be ideals such that $I \subseteq J$. Then, for any $x \in A$, if I is complete for x then also J is.*

Proof. Since I is a subset of J, (I, x) is a subset of (J, x) and hence if (J, x) is consistent then also (I, x) is consistent. Assume now that (J, x) is consistent in order to show that $x \in J$. Then (I, x) is consistent and hence, by completeness of I for x, we get $x \in I$. Now $x \in J$ follows from $I \subseteq J$. □

We are now ready to prove the main theorem of this section.

Theorem 3.4 (The countable maximal ideal theorem). *Let $\mathscr{A}$ be a countable ring and suppose that I is one of its consistent ideals. Then there is a maximal ideal M in $\mathscr{A}$ such that $I \subseteq M$.*

Proof. Let x_n be any surjective enumeration of A and let us define a countable chain $(M_n)_{n \in \omega}$ of ideals. To begin with, set $M_0 = I$. Suppose now that we have already defined M_n and define $M_{n+1} \equiv (X_{n+1})$ as the minimal ideal containing the set

$$X_{n+1} = \{x \mid (x \in M_n) \vee ((M_n, x) \text{ is consistent } \& \ x = x_n)\}.$$

Finally, set $M \equiv \bigcup_{n \in \omega} M_n$. Note that M, being the union of a chain of ideals, is also an ideal. Moreover, $I \subseteq M$ trivially holds. So we have only to prove that M is maximal.

First observe that, for any $n \in \omega$, an element m of M_{n+1} is equal to $a_1 y_1 + \cdots + a_k y_k$, for some $y_1, \ldots, y_k \in X_{n+1}$ and $a_1, \ldots, a_k \in A$. Then, by induction over k, it is possible to see that either $m \in M_n$, or it has the form $m_1 + a x_n$, for some $m_1 \in M_n$ and $a \in A$; and in the latter case $M_{n+1} = (M_n, x_n)$ and (M_n, x_n) is consistent.

Now, we prove by induction on n that, for any $n > 0$, M_n is both consistent and complete for x_{n-1}.

This result holds by hypothesis if $n = 0$ since $M_0 = I$. Then, let us consider the case n is a successor.

Consistency: We have to prove that $1_A \notin M_{n+1}$. Assume $1_A \in M_{n+1}$, in order to obtain a contradiction. Then either 1_A is in M_n, and we found a contradiction, or $1_A = m_1 + a x_n$ for some $m_1 \in M_n$ and $a \in A$, and (M_n, x_n) is consistent. But $1_A = m_1 + a \cdot x_n$ means that (M_n, x_n) is inconsistent and so we found a contradiction.

Completeness: We have to prove that M_{n+1} is complete for x_n. Assume (M_{n+1}, x_n) is consistent. Then (M_n, x_n) is consistent since it is a subset of (M_{n+1}, x_n). Thus, by definition of M_{n+1}, we conclude $x_n \in M_{n+1}$.

Thus, we checked that all the ideals M_n are consistent. Therefore, also M is consistent, being the union of a chain of consistent ideals. Finally, by monotonicity, M is complete over each x_n, since it includes M_{n+1} which is complete over x_n. □

So, M is a complete consistent ideal, that is, a maximal extension of I.

3.2. Some comments on the result

Also here, like in the previous cases, membership to M is a negated predicate. Thus, also the last construction is non-informative. We proved no more than the maximal ideal M exists, and that we can use such an existence in constructive reasoning. But from the statement $x \in M$ we get no witness. And no extra information about M is available.

The previous result can be easily strengthened to: “every ideals can be extended to a complete ideal equiconsistent with it”. This yields the previous result as a particular case when the ideal is consistent. As we already pointed out in the case of filters, this result is more interesting from a constructive viewpoint, since “inconsistency” can be a positive information. To prove this new result we only have to replace, in the previous proof, all conditions of the form “(J,x) is consistent” with the weaker condition: “if (J,x) is inconsistent, then also J is”. Such replacement is routine, and we decided to omit all details.

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