Applied Logic	**Lecture 3**
CS 486 Spring 2005	**Tuesday, February 1, 2005**

Last time, we defined the notion of a formula φ being true under an interpretation v_0, written $v_0 \models \varphi$.

Consider some examples:

- $p \wedge q$: If v_0 is the interpretation $v_0(p) = t$, $v_0(q) = t$, then $v_0 \models p \wedge q$.
- $(p \wedge q) \Rightarrow p$: This is the formula that everyone, two lectures asgo, agree was true, despite now knowing what a ”foo” or a ”bar” was. Interestingly, we can check that for any interpretation v_0, $v_0 \models (p \wedge q) \Rightarrow p$.

Definition 1 *φ is* valid *(or logically true, or true by virtue of logic, or a tautology) if $v_0 \models \varphi$ for all interpretations v_0. We typically write this as $\models \varphi$.*

Thus,$\models (p \wedge q) \Rightarrow p$ is valid. In fact, more generally, for any φ and ψ, $\models (\varphi \wedge \psi() \Rightarrow \varphi$.

How do you determine if a formula φ is valid? We need to check it is true under every interpretation. In fact, from Proposition 11 last class, it suffices to check that φ is true under every interpretation that assigns a different truth values to the propositional variables that occur in φ. How many of those are there? If there are N distinct propositional variables occuring in φ, then there are 2^N different interpretations to consider (since each variable can get one of two truth values). That's a lot, but at least, it's a finite number, and it is possible to simply enumerate all 2^N interpretations and check for each that φ is true under each interpretation. Thus, we have established:

Proposition 2 *Deciding whether a formula φ of propositional logic is a valid formula is decidable.*

(In later courses in complexity theory and algorithms, you will learn that the problem is in fact coNP-complete, that is, it is widely believed that we cannot do much better than checking all 2^N interpretations.)

A number of notations are associated with validity. A formula φ is a *contradiction* (or logically false) if $v_0 \not\models \varphi$ for all interpretations φ. A formula φ is *satisfiable* if there exists an interpretation v_0 such that $v_0 \models \varphi$.

Proposition 3 *Let φ be a formula. The following are equivalent:*

(1) φ is valid;

(2) $\neg\varphi$ is a contradiction;

(3) $\neg\varphi$ is not satisfiable.

Proof. Immediate from the definitions. Show that (1) implies (2) implies (3) implies (1). $\square$

We now define a new form of formula, by abbreviation. Define $\varphi \Leftrightarrow \psi$ as an abbreviation for $(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$. It is easy to check that $v_0 \models \varphi \Leftrightarrow \psi$ if and only if $v_0 \models \varphi$ exactly when $v_0 \models \psi$. (They are both true or they are both false.) If φ and ψ have the same truth value under all interpretations, they are, in some sense, equivalent.

Definition 4 *Two formulas φ and ψ are said to be* logically equivalent *if $\models \varphi \Leftrightarrow \psi$.*

Proposition 5 *The following equivalences hold:*

- $\models (\varphi \wedge \psi) \Leftrightarrow (\psi \wedge \varphi)$ *(commutativity of conjunction)*
- $\models (\varphi \vee \psi) \Leftrightarrow (\psi \vee \varphi)$ *(commutativity of disjunction)*
- $\models (\neg\neg\varphi) \Leftrightarrow \varphi$ *(negation elimination)*
- $\models (\varphi \wedge \psi) \Leftrightarrow \neg(\neg\varphi \vee \neg\psi)$ *(de Morgan duality for $\wedge$)*
- $\models (\varphi \vee \psi) \Leftrightarrow \neg(\neg\varphi \wedge \neg\psi)$ *(de Morgan duality for $\vee$)*
- $\models (\varphi \Rightarrow \psi) \Leftrightarrow (\neg\varphi \vee \psi)$

Proof. Explicit verification. $\square$

Intuitively, one should be able to replace a formula by another logically equivalent formula without affecting truth value. If φ, ψ and γ are formula, let $\varphi[\psi \mapsto \gamma]$ the formula obtained by replacing every occurrence of ψ as a subformula of φ by γ.

Proposition 6 *Let v_0 be an interpretation, and let ψ and γ be logically equivalent formulas (i.e., $\models \psi \Leftrightarrow \gamma$). The following holds: $v_0 \models \varphi$ if and only if $v_0 \models \varphi[\psi \mapsto \gamma]$.*

Proof. By induction on formulas. $\square$

A consequence of this proposition is that whenever $\models \psi \Leftrightarrow \gamma$, then $\models \varphi \Leftrightarrow \varphi[\psi \mapsto \gamma]$.

By Proposition 6, we can use the equivalences of Proposition 5 to eliminate all occurrences of $\Rightarrow$ (by translating them to $\vee$) and all occurrences of $\vee$ (by translating them to $\wedge$), and obtain a logically equivalent formula. Thus, in a precise sense, we could have defined propositional logic taking only $\neg$ and $\wedge$ as primitives, deriving the others by abbreviations, and still be able to write all the same formulas with the same truth values as with the "full" logic. Alternatively, we could have taken only $\neg$ and $\vee$ as primitives, or $\neg$ and $\vee$ as primitives. (Why?) In fact, it turns out that one can get away with only a single binarsy operator. (See Smullyan, p.14, Exercise 5).