4 May 2026

Sparse Recovery (part 2)
Community Detection

Announcements

1. Prob Set 4 due tomorrow (May 5), 11:59 pm.
   (or ask for extension)
2. Final Exam
   - Sunday, May 10, 7:00pm, Gates G01.
   - One-page (double sided) handwritten (by you) study sheet allowed.
   - Practice problems in pinned Ed post.
3. 5850 Final Project
   - Due May 10, 11:59pm (or ask for extension)
   - Turn in .zip file on Canvas.
   - Work in groups of 1-2 people
   - Assignment description in pinned Ed post.
4. Please fill out a Course Evaluation!
   (It constitutes 2% of your grade.)

RECAP.

Sparse Recovery: design measurement matrix $A \in \mathbb{R}^{m \times n}$ s.t. any s-sparse vector $x \in \mathbb{R}^n$ can be recovered from $b = Ax \in \mathbb{R}^m$.

Goal. When $s \ll n$, should be possible to use $m \ll n$ measurements, and an efficient algorithm, to find $\vec{x}$.

Recall. If $m > 2s$ and entries of A are iid $N(0,1)$ then for any s-sparse $x_0 \in \mathbb{R}^n$, the unique solution to $\{ Ax = b := Ax_0,\ x \text{ s-sparse} \}$ is $x = x_0$.

Problem. We don't believe $\exists$ efficient algo to find x when $m = 2s+1$, say.

Good news Candes & Tao showed you can find s-sparse x satisfying $Ax = b$ (if one exists) efficiently using

$$\min \|x\|_1$$
$$\text{s.t. } Ax = b.$$

when A satisfies the $(3s)$-restricted isometry property with constant $\varepsilon < \frac{1}{3}$.

Better news. $A \in \mathbb{R}^{m \times n}$ with $N(0, \frac{1}{m})$ entries satisfies $(3s)$-RIP with const. ε when

$$m \geq \frac{50}{\varepsilon^2}\left(3s \log(n) + \log\left(\frac{2}{\delta}\right)\right)$$

with probability $1-\delta$.

Gaussian hash lemma.

Recall. A satisfies s-RIP with const ε if

$$(1-\varepsilon)\|x\|^2 \leq \|Ax\|^2 \leq (1+\varepsilon)\cdot\|x\|^2$$

$\forall$ s-sparse x.

Why does L_1-minimization work for RIP matrices?

We need to show $(3s)$-RIP with const $\varepsilon < \frac{1}{3}$ implies...

$$\nexists \; x_1 \neq x_0 \quad \text{s.t.} \quad \|x_1\|_1 < \|x_0\|_1,$$

$$Ax_1 = Ax_0 = b$$

x_0 s-sparse.

$x = x_0 - x_1$. $\quad Ax = b - b = 0$

$\|x - x_0\|_1 = \|-x_1\|_1 = \|x_1\|_1 < \|x_0\|_1$

$\Leftarrow$ x mostly s-sparse, $Ax = 0$.

<u>Def.</u> $x \in \mathbb{R}^n$ is <u>mostly s-sparse</u> if $\exists$ s-sparse $x_0 \in \mathbb{R}^n$ with

$$\|x_0\|_1 > \|x - x_0\|_1.$$

<u>Ex</u> Is $\begin{bmatrix} 4 \\ 8 \\ 5 \\ 0 \\ 5 \\ 8 \\ 5 \\ 0 \end{bmatrix}$ mostly 2-sparse? mostly 3-sparse?

$$\begin{bmatrix} 4 \\ 8 \\ 5 \\ 0 \\ 5 \\ 8 \\ 5 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 8 \\ 0 \\ 0 \\ 0 \\ 8 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 5 \\ 0 \\ 5 \\ 0 \\ 5 \\ 0 \end{bmatrix}$$

$\|x_0\|_1 = 16 \qquad \|x - x_0\|_1 = 19$

FACT. If A is $(3s)$-RIP with const $\varepsilon < \frac{1}{3}$, and $x \in \mathbb{R}^n$ mostly s-sparse, then

$$\|Ax\|_2 > \frac{1}{2}\left(\sqrt{1-\varepsilon} - \sqrt{\frac{1+\varepsilon}{2}}\right)\|x\|_2.$$

$\varepsilon < 1/3$

$\Rightarrow 1-\varepsilon > \frac{1+\varepsilon}{2}$

$\Rightarrow \sqrt{1-\varepsilon} - \sqrt{\frac{1+\varepsilon}{2}} > 0$

__Proof.__ Sort coordinates of x, $|x_1| \geq |x_2| \geq \dots \geq |x_n|$.

Organize into blocks

$J_0 = s$ largest coords.

$J_1 = 2s$ next largest

$J_2 = 2s$ next largest after that

$\vdots$

For $i = 0, 1, 2, \dots$

$x_i = x$ with all the coordinates except J_i zeroed out.

E.g. with $s = 2$

$x = (100, 98, 86, 84, 72, 71, 64, 58, 32, 4)$

$x_0 = (100, 98, 0, 0, 0, 0, 0, 0, 0, 0)$

$x_1 = (0, 0, 86, 84, 72, 71, 0, 0, 0, 0)$

$x_2 = (0, 0, 0, 0, 0, 0, 64, 58, 32, 4)$

Mostly s-sparse means:

$$\|x_0\|_1 > \|x_1\|_1 + \|x_2\|_1 + \dots + \|x_k\|_1.$$

Useful fact: $\|x_{i+1}\|_\infty \leq \frac{1}{2s}\|x_i\|_1.$

$$\frac{1}{\sqrt{2s}}\|x_{i+1}\|_2 \leq$$

$$\|x_{i+1}\|_2 \leq \frac{1}{\sqrt{2s}}\|x_i\|_1$$

$$Ax = A(x_0+x_1) + Ax_2 + Ax_3 + \dots + Ax_k$$

$$\|Ax\|_2 \geq \|A(x_0+x_1)\|_2 - \|Ax_2\|_2 - \|Ax_3\|_2 - \dots - \|Ax_k\|_2$$

(triangle ineq)

$$\geq \sqrt{1-\varepsilon}\,\|x_0+x_1\|_2 - \sqrt{1+\varepsilon}\left(\|x_2\|_2 + \|x_3\|_2 + \dots + \|x_k\|_2\right)$$

(RIP)

$$\geq \sqrt{1-\varepsilon}\,\|x_0\|_2 - \sqrt{\frac{1+\varepsilon}{2s}}\left(\|x_1\|_1 + \|x_2\|_1 + \dots + \|x_{k-1}\|_1\right)$$

(mostly s-sparse)

$$\geq \sqrt{1-\varepsilon}\,\|x_0\|_2 - \sqrt{\frac{1+\varepsilon}{2s}}\,\|x_0\|_1$$

$\|x_0\|_2 \geq \frac{1}{\sqrt{s}}\|x_0\|_1$ (Cauchy-Schwarz)

$$\geq \sqrt{1-\varepsilon}\,\|x_0\|_2 - \sqrt{\frac{1+\varepsilon}{2}}\,\|x_0\|_2$$

$$= \left(\sqrt{1-\varepsilon} - \sqrt{\frac{1+\varepsilon}{2}}\right)\|x_0\|_2 \geq \frac{1}{2}\left(\sqrt{1-\varepsilon} - \sqrt{\frac{1+\varepsilon}{2}}\right)\|x\|_2$$

(proof omitted)