29 Apr 2026

Sparse Recovery.

<u>Def.</u> $x \in \mathbb{R}^n$ is s-sparse if it has $\leq s$ non-zero components.

<u>Problem.</u> Given matrix $A \in \mathbb{R}^{m \times n}$, vector $b \in \mathbb{R}^m$, sparsity bound $s \in \mathbb{N}$ find a s-sparse solution of $Ax = b$ if one exists.

E.g. decomposing sound wave into $\leq s$ frequencies

decoding positions of stars from space telescope raw data.

statisticians searching for linear relationship among small subset of n variables.

<u>Themes.</u> (1) Sparsity as a way to circumvent "curse of dimensionality."

"You can search in a very high dimension for a very sparse vector and still hope to find it."

(2) Random matrices enable efficient algo's.

(3) Statistical vs. computational tradeoffs.

The minimum amt of data that enables a problem to be solved is much less than the amt. required to solve efficiently.

m = "amount of data"

Problem. Given matrix $A \in \mathbb{R}^{m \times n}$, vector $b \in \mathbb{R}^m$, sparsity bound $s \in \mathbb{N}$ find a s-sparse solution of $Ax = b$ if one exists.

Answer Ø. If $m = n$ and A invertible then $x = A^{-1}b$.

Answer 1. If $m < s$ then multiple s-sparse solutions are guaranteed to exist.

Even if we commit to sparse vectors of the form $(x_1, \ldots, x_s, 0, 0, 0, \ldots, 0) \in \mathbb{R}^n$, then

$$\left[\, A_s \;\middle|\; A_{[n] \setminus s} \,\right] \begin{bmatrix} \vec{x}_s \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = A_s \vec{x}_s$$

$<s$ rows, s columns

Finding $\vec{x}_s$ amounts to solving $A_s \vec{x}_s = b$. If any solutions exist, $\exists$ infinitely many solutions because A_s has positive dimensional nullspace.

Answer 2. If A has $m \geq 2s$ rows and there is no linear dependency among any $2s$ columns of A

then $Ax = b$ cannot have more than one s-sparse solution.

<u>Proof.</u> Suppose x_1 and x_2 both s-sparse. Then $A(x_1 - x_2)$ is a linear combination of $\leq 2s$ columns of A.

$\Rightarrow$ either $x_1 = x_2$ or

$$A(x_1 - x_2) \neq 0$$

In latter case Ax_1 and Ax_2 can't both equal b.

$\Rightarrow$ $Ax = b$ has at most 1 s-sparse solution.

<u>Remark.</u> If entries of A are i.i.d. $N(0,1)$ and $m \geq 2s$

$$\Pr(\exists\ 2s \text{ or fewer lin dep't cols}) = 0$$

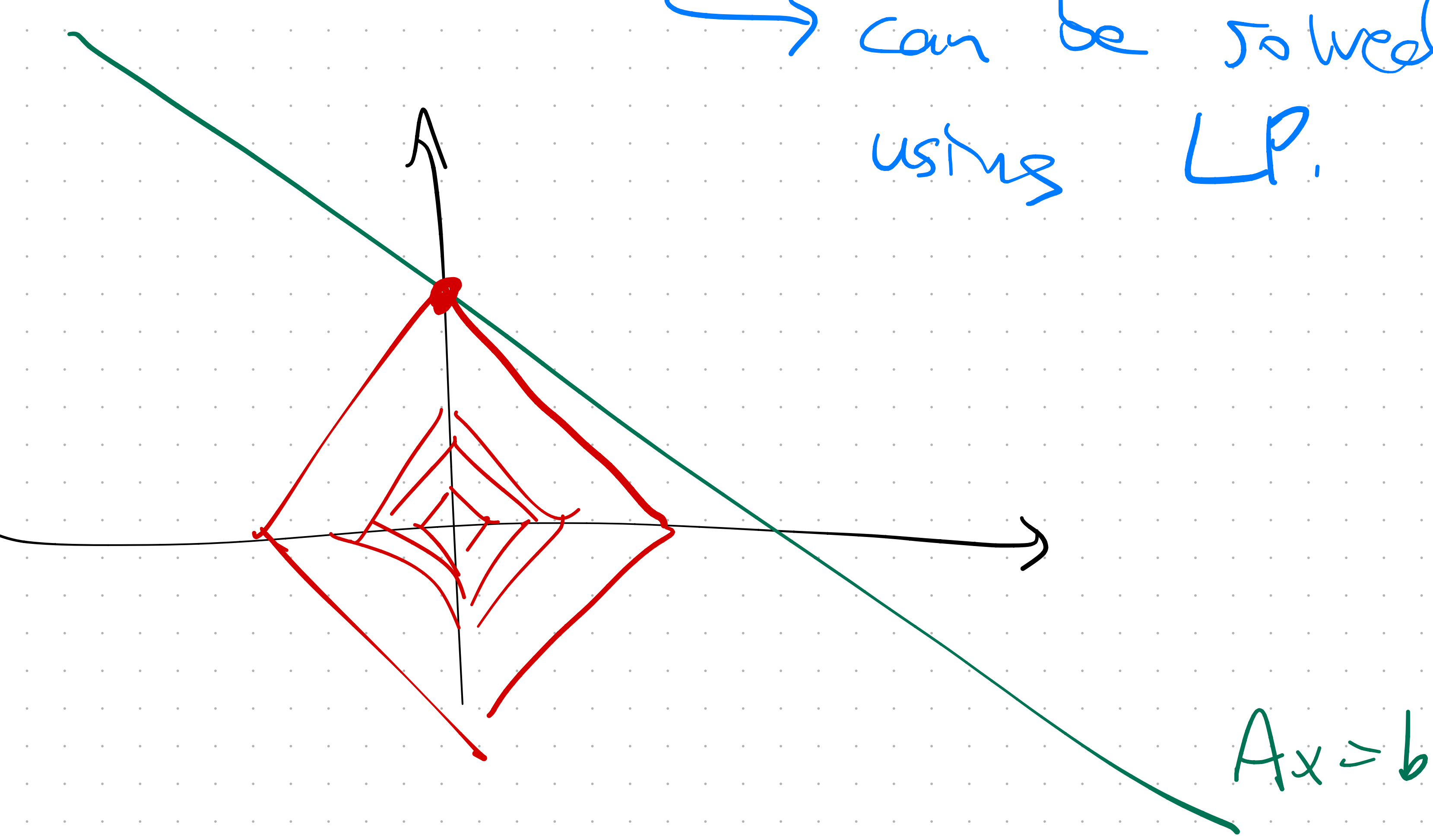
<u>Conclusion</u>, If A drawn from this distribution, and x is s-sparse, then x is the unique s-sparse solution to $Ax = b$.

<u>Problem.</u> For $A \in \mathbb{R}^{2s \times n}$ no efficient algorithm to find x is known.

<u>We'll see.</u> For $A \in \mathbb{R}^{O(s \log n) \times n}$ you can find x w.h.p. by solving

$$\left[\begin{array}{ll} \min & \|x\|_1 \\ \text{s.t.} & Ax = b. \end{array}\right]$$

↪ can be solved efficiently using LP.



"Minimize ℓ_1-norm subj to constraints tends to promote sparsity"

Def. $A \in \mathbb{R}^{m \times n}$ satisfies s-restricted isometry property (s-RIP) with const $\varepsilon_s > 0$ if the following is satisfied by every s-sparse x:

$$(1-\varepsilon_s)\|x\|^2 \leq \|Ax\|^2 \leq (1+\varepsilon_s)\|x\|^2.$$

Compare with Gaussian Hashing Lemma:

$w \in W$ (d-dimensional)

$$(1-2\varepsilon)\|w\| \leq \|Aw\| \leq (1+2\varepsilon)\|w\|.$$

$$\Updownarrow$$

$$(1-2\varepsilon)^2\|w\|^2 \leq \|Aw\|^2 \leq (1+2\varepsilon)^2\|w\|^2$$

FACT. $A \in \mathbb{R}^{m \times n}$ with $N(0, \frac{1}{n})$ iid entries and

$$m \geq \frac{50}{\varepsilon_s^2}\left(s \ln(n) + \ln\left(\frac{2}{\delta}\right)\right)$$

then w. prob $> 1-\delta$

A satisfies s-RIP with

const ε_s.

Proof. Gaussian Hash Lemma plus union bound.

{ s-sparse vectors }

= union of $\binom{n}{s}$ subspaces each of dimension s.

Call them $W_1, \ldots, W_{\binom{n}{s}}$.