22 Apr 2026 — Gaussian vectors & matrices

Recall. For $\mu \in \mathbb{R}^n$ and $\Sigma \in \mathbb{R}^{n \times n}$ posi def. the multivariate Gaussian $N(\mu, \Sigma)$ is the distrib with density

$$f(x) = \frac{1}{\sqrt{\det(\Sigma)}} \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)}.$$

E.g. if $\mathbb{1} \in \mathbb{R}^{n \times n}$ denotes identity matrix $N(0, \mathbb{1})$ has density

$$\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}x^T x} = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 - \dots - \frac{1}{2}x_n^2}$$

$$= \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_i^2}\right)$$

$N(0, \mathbb{1})$ is the distrib of n i.i.d. random variables, each $N(0,1)$ distributed.

Suppose $B \in \mathbb{R}^{n \times n}$ invertible and we sample $X \sim N(0, \mathbb{1})$ and output $Y = BX + \mu$.

$$\mathbb{E}[Y] = B\,\mathbb{E}[X] + \mu$$

$$\mathrm{Covar}(Y) = \mathbb{E}\left[(Y-\mu)(Y-\mu)^T\right]$$

$$= \mathbb{E}\left[(BX)(BX)^T\right]$$

$$= \mathbb{E}\left[B X X^T B^T\right]$$

$$= B \cdot \mathbb{E}[XX^T] \cdot B^T$$

$\Leftarrow \mathbb{1}$

$$= BB^T$$

Density of Y : Prob. of Y landing in box of side length ε near $y \in \mathbb{R}^n$

$$X = B^{-1}(Y - \mu).$$

Prob of X landing in $B^{-1}(\text{box})$ which has volume $\det(B^{-1}) \cdot \varepsilon^n$ and is located near $B^{-1}(y-\mu)$.

$$f_Y(y) = f_X\left(B^{-1}(y-\mu)\right) \cdot \det(B^{-1})$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\left(B^{-1}(y-\mu)\right)^T \left(B^{-1}(y-\mu)\right)} \det(B^{-1})$$

$$= \frac{1}{\det(B)} \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}(y-\mu)^T \boxed{(B^{-1})^T B^{-1}} (y-\mu)}$$

$= \Sigma^{-1}$

$$\frac{1}{\det(B)} = \frac{1}{\sqrt{\det(\Sigma)}}$$

$$\Sigma = BB^T$$

$$\det(\Sigma) = \det(B)^2$$

$$\Sigma^{-1} = (B^{-1})^T B^{-1}$$

A linear transformation applied to a Gaussian remains Gaussian.

Take $X \sim N(0, \mathbb{1})$. Set $Y = AX + \mu$.

Assume $A \in \mathbb{R}^{m \times n}$, and $\operatorname{rank}(A) = m$.

<u>Claim.</u> Y has a Gaussian distrib.

$$Y \sim N(\mu, AA^T)$$

Why? Use SVD to write

$$A = \underset{\text{orthog}}{U}\, D\, \underset{\text{orthog.}}{V^T}$$

$$D = \left(\begin{array}{ccc|c} \neq 0 & & 0 & 0 \\ & \neq 0 & & 0 \\ 0 & & \neq 0 & 0 \end{array}\right), \qquad D_0 = \begin{pmatrix} \neq 0 & & 0 \\ & \neq 0 & \\ 0 & & \neq 0 \end{pmatrix}$$

$$Y = UDV^T X + \mu.$$

$$\overset{\text{same distrib}}{\approx} \underbrace{\boxed{UD_0}}_{\substack{\| \\ B,\ \text{invertible } m\times m \text{ matrix.}}} \cdot (\text{first } m \text{ coord of } X) + \mu.$$

E.g. If X, Y are indep $N(0,1)$ what is the distrib of $3X+4Y$?

$N(0,25)$.

If $\begin{bmatrix} X \\ Y \end{bmatrix}$ is $N(0, I)$

what is distrib of $\underbrace{\begin{bmatrix} 3 & 4 \end{bmatrix}}_{A} \begin{bmatrix} X \\ Y \end{bmatrix}$?

$N(0, AA^T)$

$\begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 25 \end{bmatrix}$.

Matrices with iid Gaussian entries

Theorem. (Davidson–Szarek, Vershynin)

If A is a random matrix in $\mathbb{R}^{m \times n}$, $m \leq n$, with independent $N(0, \frac{1}{n})$ entries

then the singular values

$$\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_m(A)$$

satisfy

$$1 - \varepsilon - \sqrt{\frac{m}{n}} \leq \sigma_m(A) \leq \sigma_1(A) \leq 1 + \varepsilon + \sqrt{\frac{m}{n}}$$

with probability $\geq 1 - 2e^{-\frac{1}{2}\varepsilon^2 n}$

"Chernoff type concentration"

$\sigma_1(A)^2, \ldots, \sigma_m(A)^2$ are

the eigenvalues of AA^T.

Theorem is saying AA^T is

really close to $\mathbb{1}$.

The i,j entry of AA^T

ith row $\qquad$ jth row

is $\quad a_i \cdot a_j$.

What's $\mathbb{E}[AA^T]$? $\quad \mathbb{1}$

$i \neq j \qquad a_i, a_j \sim N(0, \frac{\mathbb{1}}{n})$ indep.

$$\mathbb{E}[a_i \cdot a_j] = \mathbb{E}[a_i] \cdot \mathbb{E}[a_j]$$

$$= 0.$$

$i = j \qquad \mathbb{E}[a_i \cdot a_i] = \sum_{k=1}^{n} \mathbb{E}[a_{ik}^2]$

$$= \sum_{k=1}^{n} \mathrm{Var}(a_{ik})$$

$$= 1.$$