15 Apr 2026 — Gaussians!

Continuous distributions on $\mathbb{R}$ can be encoded by their CDF

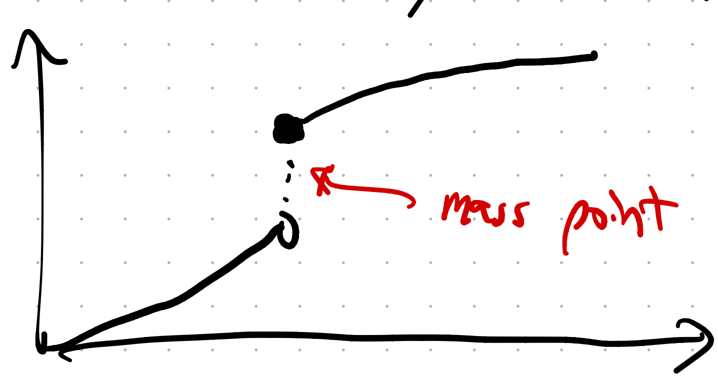
random variable — threshold.

$$F(t) = \Pr(X \le t)$$

F is always

① non-decreasing

② right-continuous, $F(t) = \lim_{t' \searrow t} F(t')$.



$\Pr(X = t) > 0.$

When F is differentiable its derivative $f(t)$ is called the <u>probability</u> <u>density</u> of X.

$f(t) \ge 0$, not necessarily ≤ 1.

$$\int_{-\infty}^{\infty} f(t)\, dt = 1.$$

The distributions we talk about will either have mass points or a density function.

Another way to think about probability densities:

The probability of $|X-t| \leq \frac{\varepsilon}{2}$

"X is in an interval of length ε centered at t"

behaves like $\varepsilon \cdot f(t) + o(\varepsilon)$ as $\varepsilon \to 0$.

Probabilities of small intervals near t

are $\varepsilon \cdot (\text{interval length}) + \text{lower-order terms.}$

$\cdot$ density $f(t)$

E.g. Suppose h is differentiable func $\mathbb{R} \to \mathbb{R}$

and X has prob. density f.

What is the density of $h(X)$

at $s = h(t)$?

$\Uparrow$

X is here with prob $\approx \varepsilon \cdot f(t)$

$\Uparrow$

$h(X)$ is here with prob $\approx \varepsilon \cdot f(t)$.

$$\text{density of } h(X) \text{ at } s = \frac{\Pr(\text{interval})}{\varepsilon \cdot \text{length}(\text{int})} = \frac{\varepsilon \cdot f(t)}{\varepsilon \cdot h'(t)} = \frac{f(t)}{h'(t)}.$$

The higher-dimensional analogue: for vector-valued random variable $\vec{X} \in \mathbb{R}^n$,

$$\Pr(\vec{X} \in B_\varepsilon) = \varepsilon \cdot f(\vec{t}) + \left(\begin{array}{c}\text{lower order}\\ \text{terms}\end{array}\right)$$

n-dimensional box of volume ε centered at $\vec{t} \in \mathbb{R}^n$

The Gaussian distribution $N(\overset{\text{mean}}{0}, \overset{\text{variance}}{1})$ is the distribution on $\mathbb{R}$ with density

$$f(t) = \boxed{\frac{1}{\sqrt{2\pi}}}\ \boxed{e^{-\frac{1}{2}t^2}}.$$

important exponential function

the necessary normalizing factor that makes $\int_{-\infty}^{\infty} f(t)\,dt = 1$.

The CDF

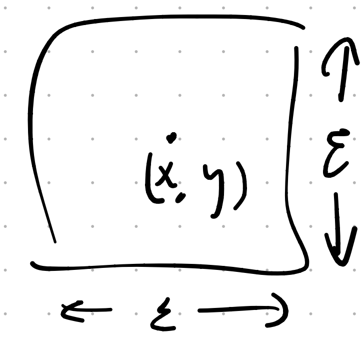
$$F(t) = \int_{-\infty}^{t} f(t)\,dt$$

cannot be written in closed form.

If X and Y are independent and both are $N(0,1)$ distributed, their joint density is

$$f(x,y) = \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\right) \cdot \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\right)$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$$



$$\varepsilon^2 \cdot f(x,y) = (\varepsilon f(x)) \cdot (\varepsilon f(y))$$

$$f(x,y) = f(x) \cdot f(y)$$

The joint distrib of (X,Y) is

(a) rotationally invariant

(b) independent.

This is what's so special about Gaussians.

Sampling from continuous distributions.

<u>Lemma</u>: If X is a rand var with continuous CDF, $F(X)$, then the rand var $Y = F(X)$ is unif distrib. in $[0,1]$.

Proof. To show: $\Pr(Y \le t) = t \quad \forall t \in (0,1)$

$$\Pr(Y \le t) = \Pr(F(X) \le t)$$
$$= \Pr(X \le F^{-1}(t))$$

maximum s s.t. $F(s) = t$

$$= F(F^{-1}(t)) = t.$$

QED.

Cor. If $Y \in [0,1]$ uniformly rand,

$X = F^{-1}(Y)$ has CDF F.

Ex. Exponential rate 1 rand var has

CDF $\qquad F(x) = 1 - e^{-x}$

density $\qquad f(x) = e^{-x} \quad (x \ge 0)$

Sampling from expt distrib.

1. Draw $Y \sim \text{unif}[0,1]$
2. $X = F^{-1}(Y) = -\ln(1-Y)$.

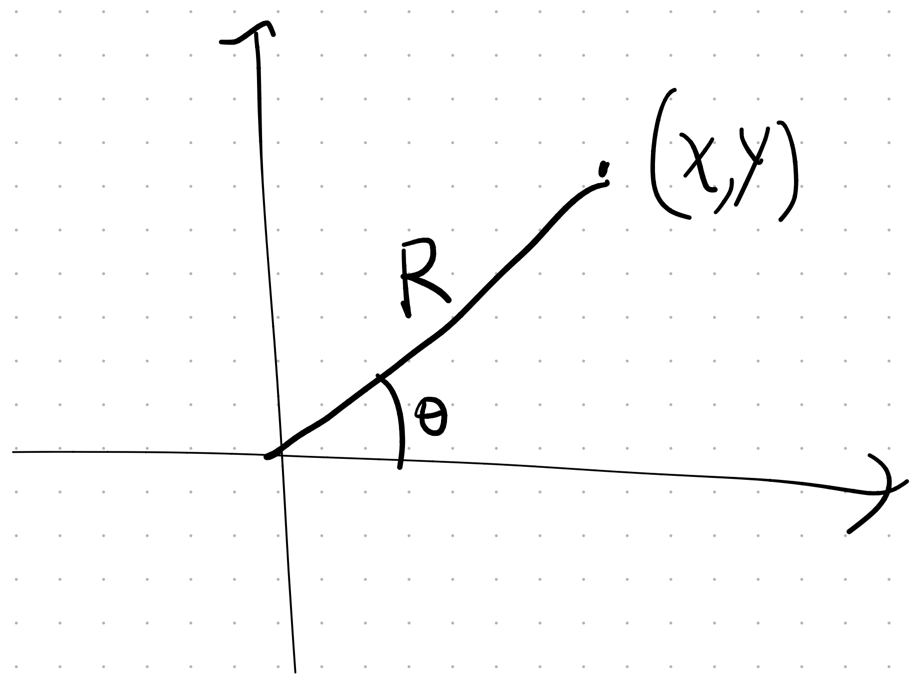
Sampling from Gaussian:

If X, Y are independent $N(0, I)$,

$$f(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}.$$

In polar coords: let $R = \sqrt{X^2+Y^2}$

$\Theta = \tan^{-1}(Y/X)$



Density in polar: $f_{polar}(r, \theta) = \frac{1}{2\pi} e^{-\frac{1}{2}r^2} \cdot r$