18 Mar 2026 Markov Chains

A Markov chain is a sequence of random variables

$$\cdots - X_0 - X_1 - X_2 - \cdots$$

such that $\forall t \; \forall x$

Necessary

Markov property

$$\Pr(X_t = x \mid \dots, X_{t-2}, X_{t-1}) = \Pr(X_t = x \mid X_{t-1}).$$

Markov Chains can:

(a) model important physical, computational, social, economic systems

(b) constitute a useful device for designing random sampling algorithms (MCMC "Markov Chain Monte Carlo")

Not part of the defin but always assumed in this class.

Time-stationary: $\forall x, y$

$$\Pr(X_{t+1} = x \mid X_t = y) = \Pr(X_t = x \mid X_{t-1} = y)$$

"Probability of transitioning from y to x at time t depends on y and x but not t."

Transition matrix $P_{yx} = \Pr(X_t = x \mid X_{t-1} = y)$.

"probability of going from y to x."

Not always assumed.

Reversibility. The probability distribution of $\ldots, X_{-1}, X_0, X_1, \ldots$ is invariant under time reversal.

For time-stationary Markov chains, we can draw their state transition graph.

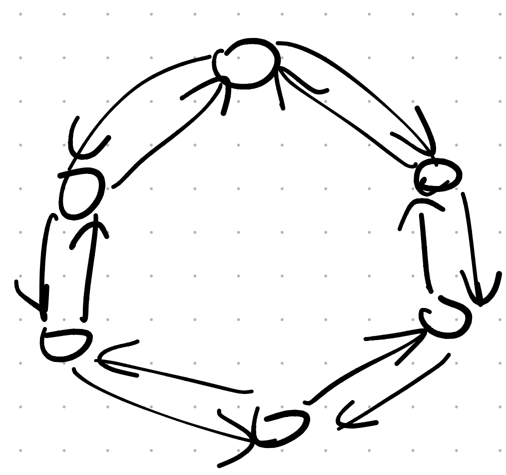
Vertices = { possible values of X_t }

(directed) edges $= \{ (x,y) \mid P_{xy} > 0 \}$

$= \{$ potential state transitions $\}$

edge labels = label (x,y) with P_{xy}.

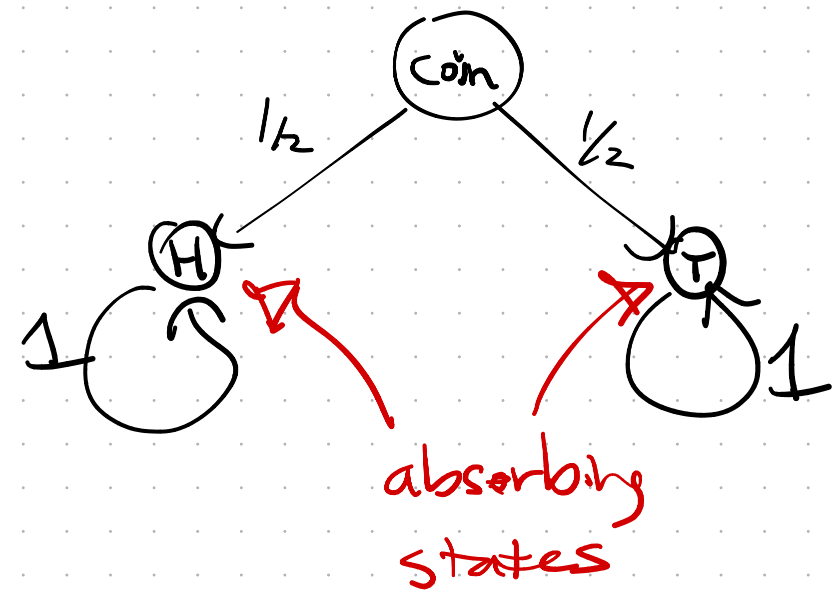
E.g.

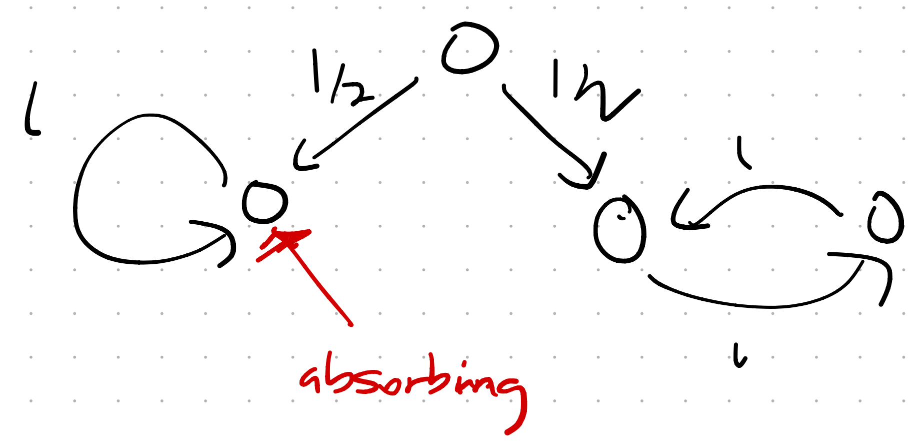


all edges labeled $1/2$.

Sum of outgoing edge labels of $v = 1$,

E.g.





Definition. A <u>stationary distribution</u> for a Markov chain is a probability distrib on states, π, such that

$$\forall x \quad \sum_y \pi_y P_{yx} = \pi_x$$

$$\pi P = \pi$$ (where π is viewed as a row vector)

"If you start out in a state drawn from π and take one step of the MC, your state is still drawn from π."

Q1. Which Markov chains have a stationary distribution?

All Markov chains with finite state set.
(And some with infinite state set....)

Q2. When is the stationary distrib unique?

Yes, when the Markov Chain is <u>irreducible</u>
(see below)

Q3. When is the distrib of X_t guaranteed to converge to π as $t \to \infty$?

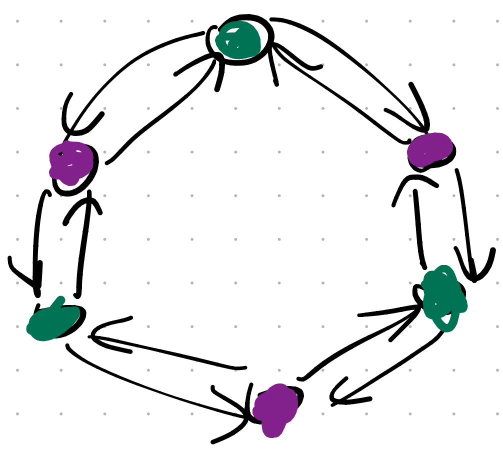
Yes, when the Markov Chain is <u>ergodic</u>.
(see below)

Def. Markov chain is <u>irreducible</u> if its state transition graph is <u>strongly connected</u>

meaning: $\forall$ pair of states x & y there is a path from x to y in the state transition graph.

Markov Chain is <u>ergodic</u> if

$\forall$ pairs of states x & y and all sufficiently large N, there is a path of length N from x to y in the state transition graph.



all edges labeled $1/2$.

Proofs of all the above properties follow from the <u>Perron-Frobenius Theorem</u>.