16 Mar 2026

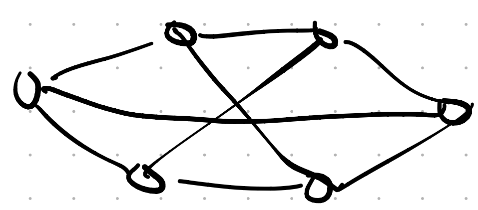
$G(n,p)$ diameter and expansion + "probabilistic method"

Def. In a graph G with vertices u, v

$$d(u,v) = \begin{cases} \text{\# edges in shortest } u\text{-}v \text{ path} \\ \infty \text{ if no } u\text{-}v \text{ path exists.} \end{cases}$$

$$\text{diameter}(G) = \max_{u,v} \{d(u,v)\}.$$

E.g.



diameter = 2

FACT. When $p = \frac{c \ln(n)}{n}$ and $c > 1$,

$\text{diameter}(G(n,p)) = O_c(\log n)$ with probability $1 - o(1)$ as $n \to \infty$.

implicit constant in $O(\cdots)$ may depend on c.

TODAY. Prove this for $c = 7$.

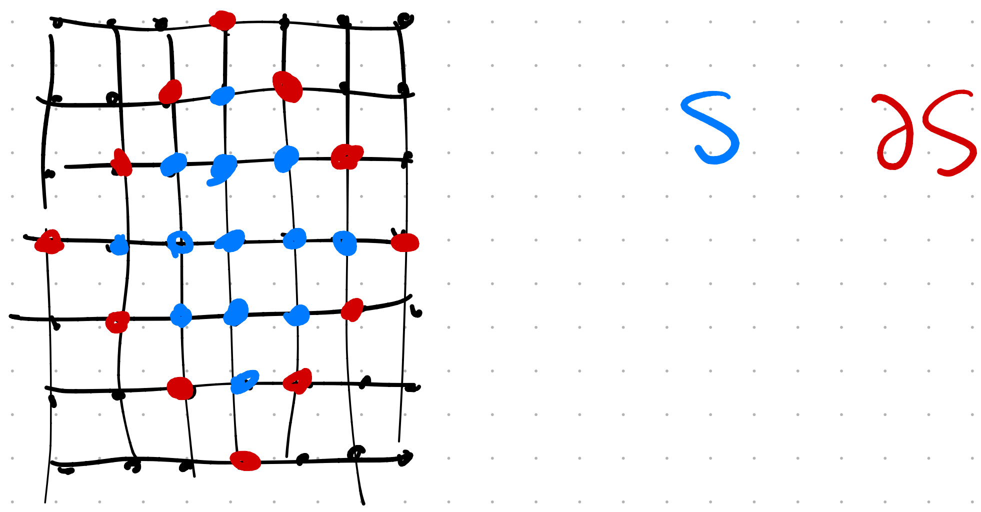
(The choice of $c=7$ is for convenience)

Def. Finite undirected graph $G = (V,E)$ with n-vertices is an α-vertex-expander if

$$\forall S \subseteq V \quad |S| \leq \frac{n}{2}, \quad |\partial S| \geq \alpha \cdot |S|$$

where $\partial S = \{v \notin S \mid \exists \text{ edge } (u,v) \text{ with } u \in S\}$.

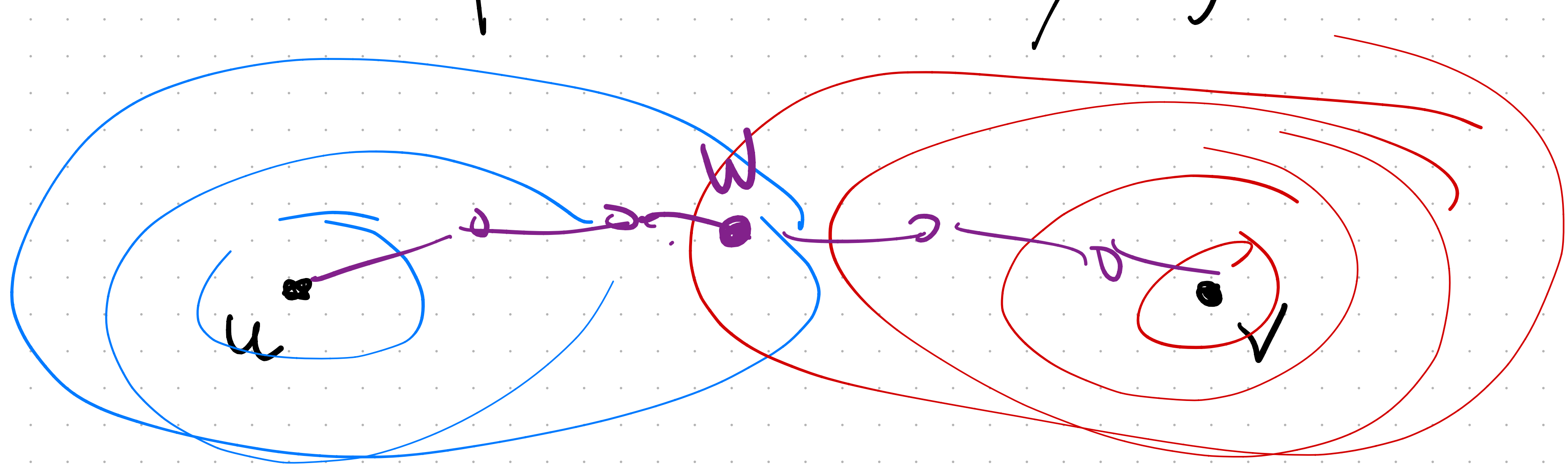
Ex.



$n \times n$ grid is not an α-expander unless $\alpha = O\left(\frac{1}{n}\right)$. $\frac{|\partial S|}{|S|} = \frac{O(n)}{\Omega(n^2)}$

Lemma. If G is an α-expander then $\text{diameter}(G) \leq O\left(\frac{\log n}{\alpha}\right)$ where $n = |V(G)|$.

Proof sketch. Want to find short u-v path for any given u, v.



Consider running BFS from u to discover "layers" defined by

$$L_0 = \{u\}$$

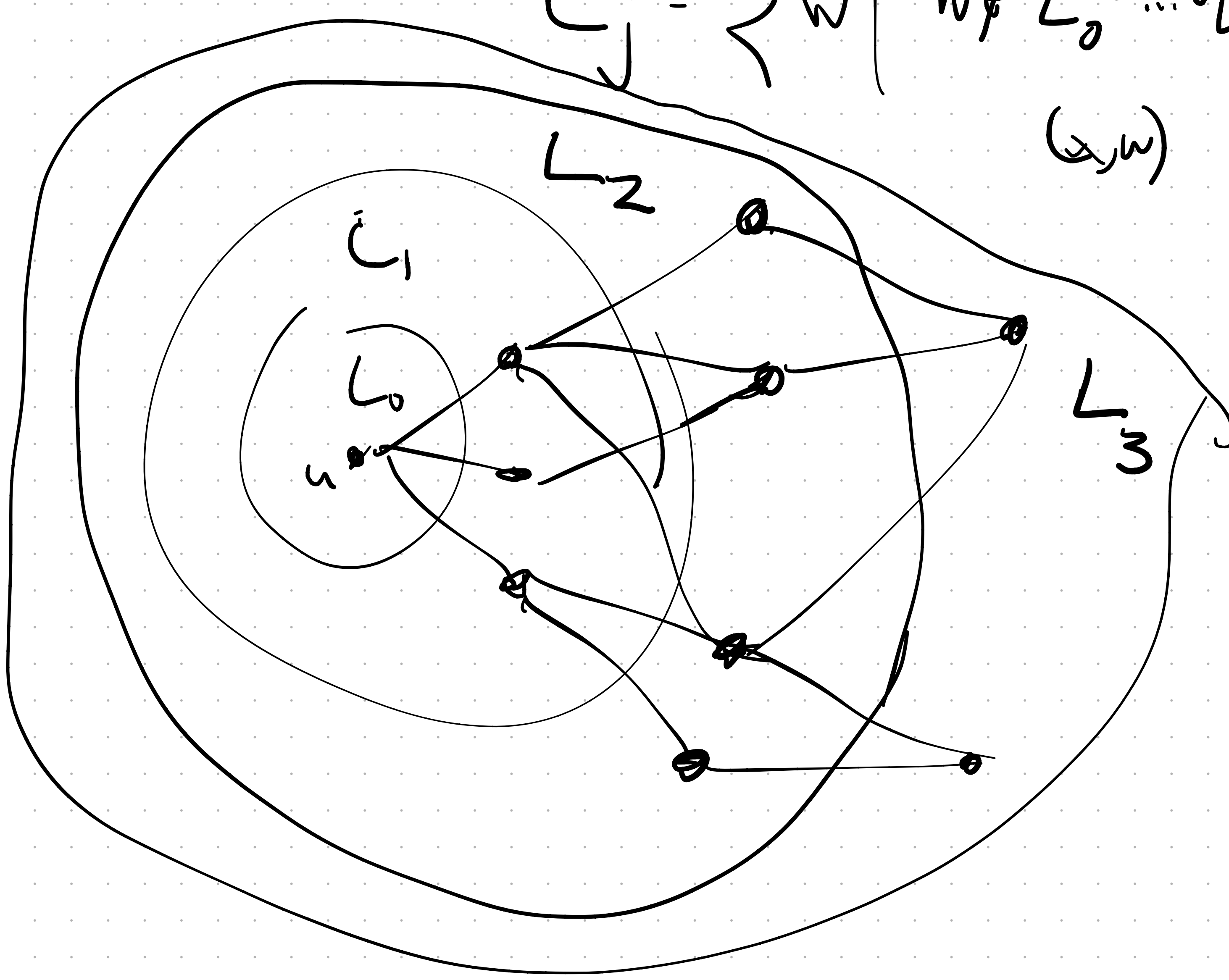
$$L_1 = \{ w \mid w \notin L_0, \ \exists \text{ edge } (x,w) \text{ with } x \in L_0 \}$$

$$L_2 = \{ w \mid w \notin L_0 \cup L_1 \ \exists \text{ edge } (x,w) \text{ with } x \in L_1 \}$$

$\vdots$

$$L_j = \{ w \mid w \notin L_0 \cup \dots \cup L_{j-1}, \ \exists \text{ edge } (x,w) \text{ with } x \in L_0 \cup \dots \cup L_{j-1} \}$$

equivalently $x \in L_{j-1}$.



What does α-expansion tell us about sizes of layers?

Let $S_j = L_0 \cup L_1 \cup \dots \cup L_j$.

If $|S_j| \le \frac{n}{2}$ then $|\partial S_j| \ge \alpha \cdot |S_j|$.

But $\partial S_j = L_{j+1}$.

So,

$$|L_{j+1}| \geq \alpha \cdot |S_j|$$

$$|S_{j+1}| = |S_j| + |L_{j+1}| \geq (1+\alpha) \cdot |S_j|.$$

From $|S_0| = 1$, $|S_{j+1}| \geq (1+\alpha) \cdot |S_j| \quad \forall j$

$$\Longrightarrow |S_j| \geq (1+\alpha)^j$$

... as long as $|S_{j-1}| \leq \frac{n}{2}$.

Conclusion: For $j \leq \left\lceil \log_{1+\alpha}\left(\frac{n+1}{2}\right) \right\rceil$.

$$|S_j| \geq (1+\alpha)^j.$$

Do the same thing with

$$T_j = \{\text{vertices reachable from } v \text{ in } j \text{ or fewer hops}\}$$

$$|T_j| \geq (1+\alpha)^j.$$

Set $j = \left\lceil \log_{1+\alpha}\left(\frac{n-1}{2}\right) \right\rceil$.

$$(1+\alpha)^j > \frac{n}{2}.$$

$$|S_j| > \frac{n}{2}, \quad |T_j| > \frac{n}{2}$$

S_j, T_j cannot be disjoint

$\implies \exists$ vertex $w \in S_j \cap T_j$

s.t. $d(u,w) \leq j$, $d(v,w) \leq j$

$$\therefore \quad d(u,v) \leq d(u,w) + d(w,v) \leq 2j$$

$$\implies \text{diameter}(G) \leq 2\left\lceil \log_{1+\alpha}\left(\frac{n+1}{2}\right)\right\rceil$$

$$= O\left(\frac{\log(n)}{\log(1+\alpha)}\right)$$

$$= O\left(\frac{\log n}{\alpha}\right).$$

When is $G(n,p)$ an expander?

Requires $p > \frac{\ln(n)}{n}$ because

disconnected graphs aren't expanders.

We'll show $p \geq 7\frac{\ln(n)}{n} \implies \frac{1}{2}$-expansion.

Say $S \subseteq T$ is a <u>bad pair</u>

if $|S| \leq \frac{n}{2}$, $\partial S \subseteq T$,

and $|T \setminus S| \leq \frac{1}{2} \cdot |S|$.

<u>Observation</u>. G is $\frac{1}{2}$-expander

iff it has no bad pair.

A bad pair with $|S| = k$,

must have $|T| = |S| + |T \setminus S|$

$$\leq \left(1 + \tfrac{1}{2}\right) \cdot k.$$

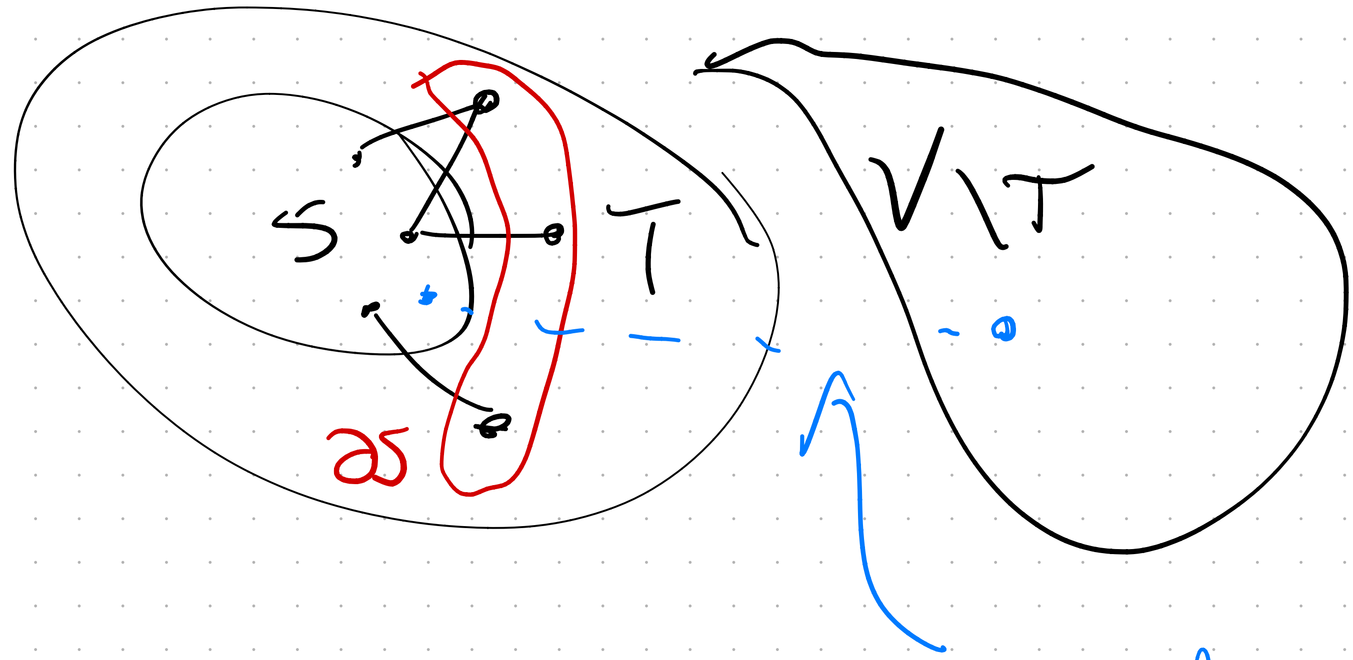
For $\alpha = \frac{1}{2}$, $k \leq \frac{n}{2}$,

$$(1+\alpha) \cdot k \leq \frac{3}{2} \cdot \frac{n}{2} = \frac{3n}{4}.$$

$$\Rightarrow |V \setminus T| = n - |T| \geq \frac{n}{4}.$$

For given $S \subseteq T$, with $k = |S| \leq \frac{n}{2}$,

$|T| \leq (1+\alpha) k$,



no edges from S to $V \setminus T$.

$|S| = k, \quad |V \setminus T| \geq \frac{n}{4}$

$\Longrightarrow$ there are $\geq \frac{kn}{4}$ pairs (u,v) with $u \in S, \; v \notin T$.

In order for S, T bad pair all these (u,v) pairs must be non-edges of $G(n,p)$.

$$\Longrightarrow \Pr(S,T \text{ bad pair}) \leq (1-p)^{nk/4}.$$

$$\mathbb{E}[\#\text{ bad pairs}] \leq \sum_{\ell=1}^{3n/4} \sum_{k=2\ell/3}^{\ell} \underbrace{\binom{n}{\ell}}_{\substack{\text{Choosing } T \\ \text{with } \ell \\ \text{elements}}} \underbrace{\binom{\ell}{k}}_{\substack{\text{Choosing} \\ S \subseteq T \\ \text{with } k \geq \frac{2\ell}{3}}} \underbrace{\boxed{(1-p)^{\frac{nk}{4}}}}_{\Pr(\text{bad})}$$

$$k \geq \frac{2\ell}{3} \implies (1-p)^{\frac{nk}{4}} \leq (1-p)^{\frac{n(2\ell/3)}{4}}$$

$$= (1-p)^{n\ell/6}$$

$$\mathbb{E}[\#\text{bad pairs}] \leq \sum_{\ell=1}^{\frac{3n}{4}} \binom{n}{\ell}(1-p)^{n\ell/6}\left[\sum_{k=2\ell/3}^{\ell}\binom{\ell}{k}\right]$$

$$\leq \sum_{\ell=1}^{3n/4} \binom{n}{\ell} 2^{\ell} (1-p)^{n\ell/6}$$

$$< -1 + \sum_{\ell=0}^{n} \binom{n}{\ell}\left[2(1-p)^{n/6}\right]^{\ell}$$

$$= -1 + \left(1 + 2(1-p)^{n/6}\right)^{n}$$

$$< -1 + \left(1 + 2e^{-pn/6}\right)^{n}$$

$$\leq -1 + \left(1 + 2n^{-7/6}\right)^{n}$$

$$p \geq \frac{7\ln(n)}{n}$$

$$\frac{pn}{6} \geq \frac{7}{6}\ln(n)$$

$$e^{-pn/6} \leq n^{-7/6}$$

$$\leq -1 + \left(e^{2n^{-7/6}}\right)^{n}$$

$$= -1 + e^{2\cdot n^{-7/6}\cdot n} = e^{2/n^{1/6}} - 1$$

$\to 0$ as $n \to \infty$

TL;DR: For $p \geq \frac{7 \ln n}{n}$,

$\Pr(\exists \text{ bad pair}) \longrightarrow 0$ as $n \to \infty$

$\Pr(G(n,p) \text{ is } \frac{1}{2}\text{-expander}) \longrightarrow 1$

When $G(n,p)$ is $\frac{1}{2}$-expander its diameter is $O(\log n)$.

This also shows that there exist graphs with n vertices and $O(n \log n)$ edges that are $\frac{1}{2}$-expanders.

(Because $\Pr(G(n,p) \text{ is } \frac{1}{2}\text{-exp}) \longrightarrow 1$

$\Pr(G(n,p) \text{ has } O(n \log n) \text{ edges}) \longrightarrow 1$

assuming $p = \frac{7 \ln n}{n}$.)

PROBABILISTIC METHOD: Show something exists because a random object has the property with high probability.