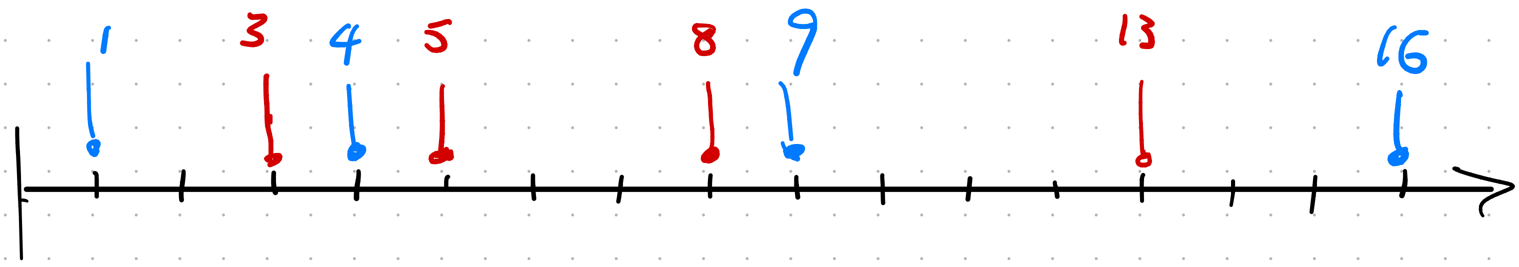
11 March 2026 — Random Graphs



Erdős – Rényi models.

$G_{n,m}$ = uniformly random sample from $\{$graphs with m edges on vertex set $[n]\}$.

invented by Erdős & Renyi

$G(n,p)$ = vertex set is $[n]$.

for each $i \neq j$ toss an independent coin with bias p.

w prob $p \Longrightarrow$ add edge $\{i,j\}$ to G

w prob $1-p \Longrightarrow$ omit it.

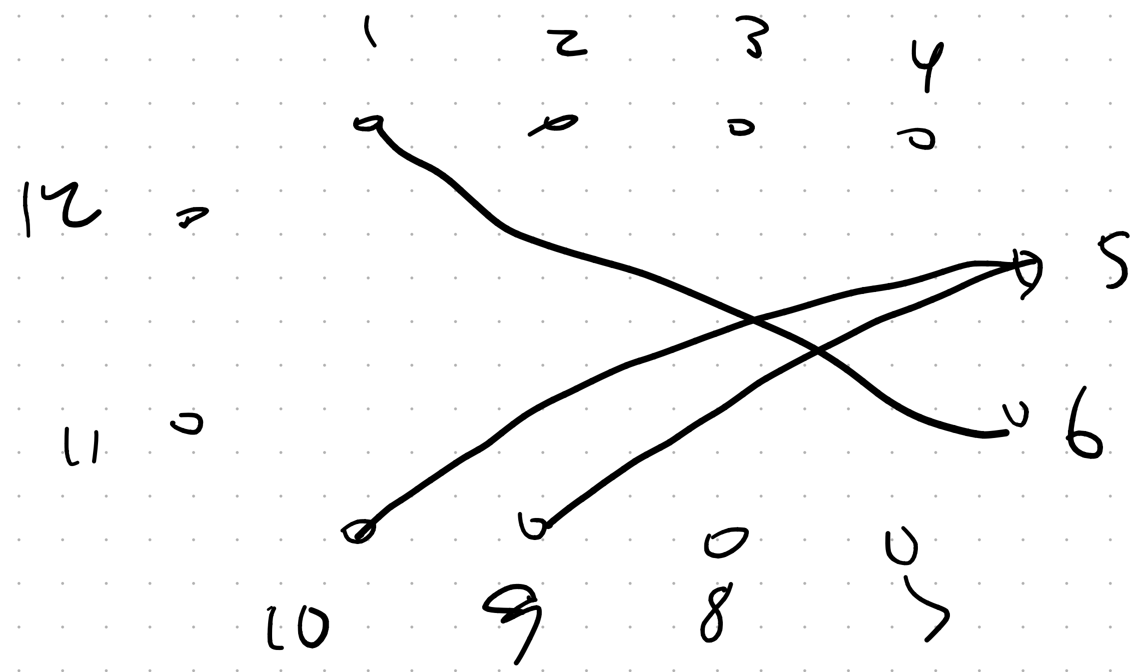
invented by Gilbert

edges is binomially distributed $B\left(\binom{n}{2}, p\right)$.

$\mathbb{E}(\#\text{ edges}) = p \cdot \binom{n}{2}$.

$G_{n,m}$ with $m = \lceil p \cdot \binom{n}{2} \rceil$ tends to behave similarly to $G(n,p)$.

This course: focus on $G(n,p)$.



Pr (first k sampled edges form a matching)

$$= 1 \cdot \frac{\binom{n-2}{2}}{\binom{n}{2}} \cdot \frac{\binom{n-4}{2}}{\binom{n}{2}} \cdots \frac{\binom{n-2k+2}{2}}{\binom{n}{2}}$$

$$= 1 \cdot \frac{n^2-5n+6}{n^2-n} \cdot \frac{n^2-9n+20}{n^2-n} \cdots$$

$$= 1 \cdot \left(1 - \frac{4n-6}{n^2-n}\right) \cdot \left(1 - \frac{8n-20}{n^2-n}\right) \cdots$$

$$< e^0 \cdot e^{-\frac{4n-6}{n^2-n}} \cdot e^{-\frac{8n-20}{n^2-n}} \cdots$$

$G(n, p)$ is probably a matching when $p \ll n^{-3/2}$

and probably not a matching when $p \gg n^{-3/2}$.

Isolated vertices?

Vertex i is isolated in $G(n,p)$ if it belongs to no edges.

By coupon collector analogy, around $m \approx \frac{n \ln n}{2}$ edges graph ceases to have isolated vertices.

Translate to $G(n,p)$ by setting $p = \frac{m}{\binom{n}{2}} \approx \frac{(n \ln n)/2}{(n^2-n)/2}$

Around $p \approx \frac{\ln(n)}{n-1}$ the random graph $G(n,p)$ ceases to have isolated vertices.

Connectivity? Diameter?

Goal for remainder of lecture:

if $p \geq \frac{3\ln(n)}{n}$ then

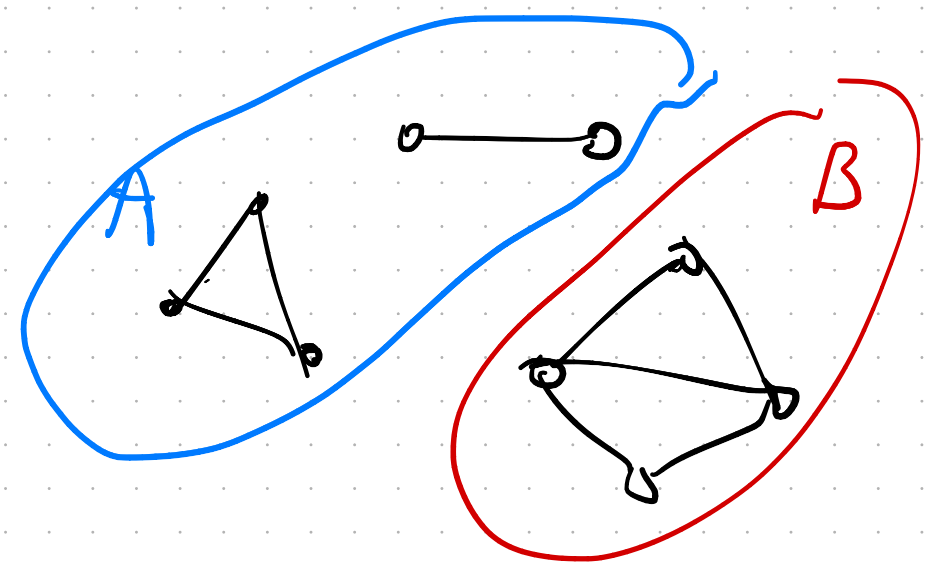
$G(n,p)$ is connected with probability $1 - o(1)$.

converging to zero as $n \to \infty$.

Known: True ~~connectivity~~ connectivity threshold

is $p \approx \frac{\ln(n)}{n}$.

Def. A separating partition of graph G is a pair of vertex sets A, B that partition $V(G)$, A, B both nonempty, and no edges from A to B.



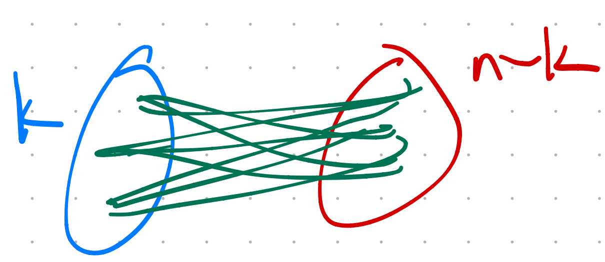
G is disconnected **iff** it has a separating partition.

$$\Pr\big(G(n,p) \text{ is disconnected}\big)$$

pairs of nonempty subsets that partition $[n]$.

$$\leq \sum_{A,B} \Pr\big(A, B \text{ forms a sep. part'n of } G(n,p)\big)$$

$$\leq \sum_{k=1}^{n/2} \binom{n}{k} \Pr\big(\text{no edges from } \{1,\dots,k\} \text{ to } \{k+1,\dots,n\}\big)$$



$n-k \geq n/2$

$$= \sum_{k=1}^{n/2} \binom{n}{k} (1-p)^{k(n-k)}$$

$(1-p)^{n-k} \leq (1-p)^{n/2}$

$$\leq \sum_{k=1}^{n/2} \binom{n}{k} (1-p)^{kn/2}$$

$$< -1 + \sum_{k=0}^{n} \binom{n}{k} (1-p)^{kn/2}$$

$$= -1 + \left[1 + (1-p)^{n/2}\right]^n$$

$p \geq \frac{3\ln(n)}{n}$

$-pn/2 \leq -\frac{3}{2}\ln(n)$

$$< -1 + \left[1 + e^{-pn/2}\right]^n$$

$$< -1 + \left[1 + n^{-3/2}\right]^n$$

$$< -1 + e^{n^{-3/2} \cdot n}$$

$$= \boxed{e^{1/\sqrt{n}} - 1} \longrightarrow 0$$

as $n \to \infty$

As promised when $p \geq \frac{3\ln(n)}{n}$

$\Pr(G(n,p) \text{ disconn}) \longrightarrow 0$ as $n \to \infty$

$\Pr(G(n,p) \text{ connected}) \longrightarrow 1$.