4 Feb 2026

The Chernoff and Hoeffding Bounds

Announcement: Problem Set 1 due Thurs 2/12
late deadline Sun 2/15 without penalty

i. Recommended group size 2-3.

ii. Each person should understand all solutions and write up in their own words.

iii. If you use GenAI on the PSet, report how you used it.

iv. Homework (20% of your grade) is 10% based on completion, 10% correctness.

Chernoff. If $X_1, \ldots, X_m$ are independent random variables taking values in $[0,1]$ and

$$X = X_1 + \cdots + X_n$$

then for all $0 \le \varepsilon \le 1$

$$\boxed{\begin{aligned} \Pr\left(X \ge (1+\varepsilon)\mathbb{E}X\right) &\le e^{-\frac{1}{3}\varepsilon^2 \cdot \mathbb{E}X} \\ \Pr\left(X \le (1-\varepsilon)\mathbb{E}X\right) &\le e^{-\frac{1}{2}\varepsilon^2 \cdot \mathbb{E}X} \end{aligned}}$$

Def. For random variable X its <u>moment</u> <u>generating</u> <u>function</u> is

$$M_X(t) = \mathbb{E}\left[e^{tX}\right]$$ (RHS could be ∞.)

Its Taylor series $M_X(t) = \sum_{n=0}^{\infty} \frac{m_n}{n!} t^n$ has coefficients

$$m_n = \mathbb{E}(X^n)$$ (n^{th} moment of X)

The <u>cumulant</u> <u>generating</u> <u>function</u> is

$$K_X(t) = \ln\left(M_X(t)\right).$$

Its Taylor series

$$K_X(t) = \sum_{n=0}^{\infty} \frac{K_n}{n!} t^n$$

the coefficients K_n are called <u>cumulants</u> of X.

$$K_1(X) = \mathbb{E}X$$

$$K_2(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2 = \mathrm{Var}(X)$$

$$K_3(X) = \mathbb{E}(X^3) - 3 \cdot \mathbb{E}(X^2) \cdot (\mathbb{E}X) + 2(\mathbb{E}X)^3$$

Lemma. If $X_1, \dots, X_n$ are independent and $X = X_1 + \dots + X_n$ then

$$M_X(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

$$K_X(t) = \sum_{i=1}^{n} K_{X_i}(t)$$

Proof. $M_X(t) = \mathbb{E}[e^{tX}] = \mathbb{E}[e^{t(X_1 + \dots + X_n)}]$

$$= \mathbb{E}[e^{tX_1} \cdot e^{tX_2} \cdot \dots \cdot e^{tX_n}]$$

by independence

$$= \prod_{i=1}^{n} \mathbb{E}[e^{tX_i}] = \prod_{i=1}^{n} M_{X_i}(t)$$

Take $\ln$ (both sides) to obtain $K_X = \sum_{i=1}^{n} K_{X_i}$

Ex. Working out $K_1, K_2, K_3(X)$.

$$M_X(t) = 1 + \overbrace{m_1 t + \frac{m_2}{2} t^2 + \frac{m_3}{6} t^3 + \dots}^{=x}$$

$$K_X(t) = \ln(\text{———— '' ————})$$

Use $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots$

$$K_X(t) = m_1 t + \frac{m_2}{2} t^2 + \frac{m_3}{6} t^3 + \dots$$

$$- \frac{1}{2}\left(m_1 t + \frac{m_2}{2} t^2 + \frac{m_3}{6} t^3 + \dots\right)^2$$

$$+ \frac{1}{3}\left(m_1 t + \frac{m_2}{2} t^2 + \frac{m_3}{6} t^3\right)^3$$

$$\dots$$

$$= \underbrace{m_1}_{= K_1} t + \underbrace{\left(\frac{m_2}{2} - \frac{m_1^2}{2}\right)}_{= \frac{K_2}{2}} t^2$$

$$+ \underbrace{\left(\frac{m_3}{6} - \frac{m_1 m_2}{2} + \frac{m_1^3}{3}\right)}_{= \frac{K_3}{6}} t^3 + \dots$$

$$K_1 = m_1 = \mathbb{E}(X)$$

$$K_2 = m_2 - m_1^2 = \mathbb{E}(X^2) - (\mathbb{E}X)^2$$

$$K_3 = m_3 - 3m_1 m_2 + 2m_1^3 = \mathbb{E}(X^3) - 3(\mathbb{E}(X^2))(\mathbb{E}X) + 2(\mathbb{E}X)^3.$$

<u>Practice question</u>: For Bernoulli RV

$$X = \begin{cases} 1 & \text{with prob } p \\ 0 & \text{with prob } 1-p \end{cases}$$

What are $M_X(t)$ and $K_X(t)$?

What about geometric

$$X = n \quad \text{w. prob.} \quad p(1-p)^{n-1}$$

for $n \in \{1, 2, \dots\}$.

For Bernoulli...

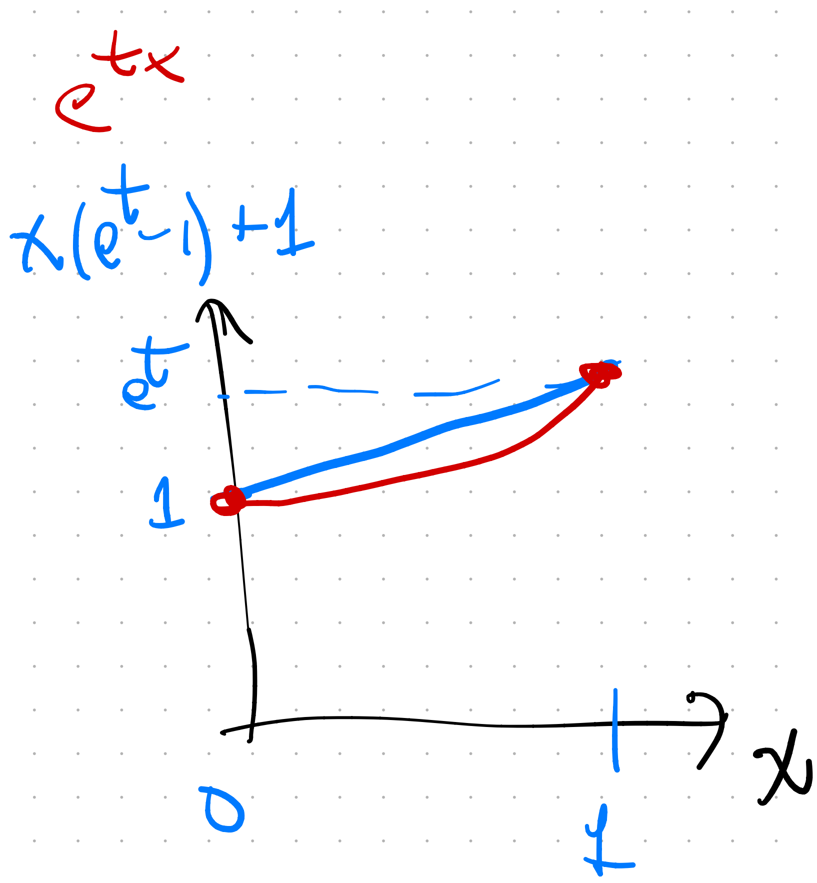
$$M_X(t) = pe^t + 1 - p$$

For X_i taking values in $[0,1]$

$$M_{X_i}(t) \leq \mathbb{E}(X_i)(e^t - 1) + 1.$$

$$\dots = pe^t + 1 - p$$

when $p = \mathbb{E}X_i$.



$$e^{tX} \leq X(e^t - 1) + 1 \quad \forall x \in [0,1]$$

$$\mathbb{E}[e^{tX}] \leq (\mathbb{E}X) \cdot (e^t - 1) + 1$$

Proof of Chernoff.

Recall $X_1, \ldots, X_n$ indep.

$$X_i \in [0,1] \quad \forall i$$

$$X = X_1 + \cdots + X_n.$$

$$M_X(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

$$\mathbb{E}[e^{tX}] \leq \prod_{i=1}^{n} \left(1 + (e^t - 1)\mathbb{E}X_i \right)$$

$$\leq \prod_{i=1}^{n} \exp\left((e^t - 1)\mathbb{E}X_i \right)$$

$$= \exp\left[\sum_{i=1}^{n} (e^t - 1)\mathbb{E}X_i \right]$$

$$\mathbb{E}[e^{tX}] \leq \exp\left((e^t - 1) \cdot \mathbb{E}X \right)$$

$$\Pr\left(X \geq (1+\varepsilon)\mathbb{E}X\right)$$

$$= \Pr\left(e^{tX} \geq e^{(1+\varepsilon)t\cdot\mathbb{E}X}\right)$$

(Markov) $$\leq \frac{\mathbb{E}[e^{tx}]}{e^{(1+\varepsilon)t\cdot\mathbb{E}X}} \leq \frac{e^{(e^t-1)\mathbb{E}X}}{e^{(1+\varepsilon)t\,\mathbb{E}X}}$$

$$= \exp\left(\left(e^t - 1 - (1+\varepsilon)t\right)\cdot\mathbb{E}X\right)$$

Minimize RHS at $t = \ln(1+\varepsilon)$,

$$e^t - 1 - (1+\varepsilon)t = \varepsilon - (1+\varepsilon)\ln(1+\varepsilon)$$

$$= \varepsilon - (1+\varepsilon)\left(\varepsilon - \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 - \dots\right)$$

$$= \cancel{\varepsilon} - \cancel{\varepsilon} - \frac{1}{2}\varepsilon^2 + \frac{1}{6}\varepsilon^3 - \dots$$

$$< -\frac{1}{3}\varepsilon^2 \quad \text{for} \quad 0 < \varepsilon < 1.$$

$$\Pr\left(X \geq (1+\varepsilon)\mathbb{E}X\right) < e^{-\frac{1}{3}\varepsilon^2\cdot\mathbb{E}X}$$