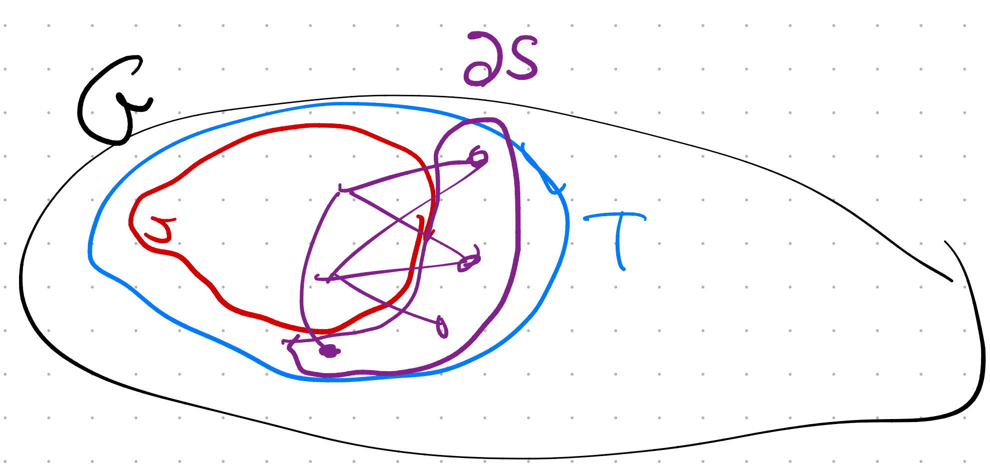
$\boxed{\text{19 Mar 2025}}$ Finishing expansion

Introducing Ramsey Theory

Theorem If $p \geq \frac{7\ln(n)}{n}$ then $G(n,p)$ is a $\frac{1}{2}$-expander with probability $\to 1$ as $n \to \infty$.

Proof. Call a pair of vertex sets S, T an $\frac{1}{2}$-expansion refuter ($\frac{1}{2}$-ER) if

- $S \subset T$
- $|S| > \frac{2}{3}|T|$, $|S| \leq \frac{n}{2}$
- All elements of ∂S belong to T.



G is not a $\frac{1}{2}$-expander

$\Updownarrow$

G has a $\frac{1}{2}$-ER.

$(\Downarrow)$ $\exists S$ $|S| \leq \frac{n}{2}$, $|\partial S| < \frac{1}{2}|S|$

$\Rightarrow$ $T = S \cup \partial S$, $|S| > \frac{2}{3}|T|$

so S, T is a $\frac{1}{2}$-ER.

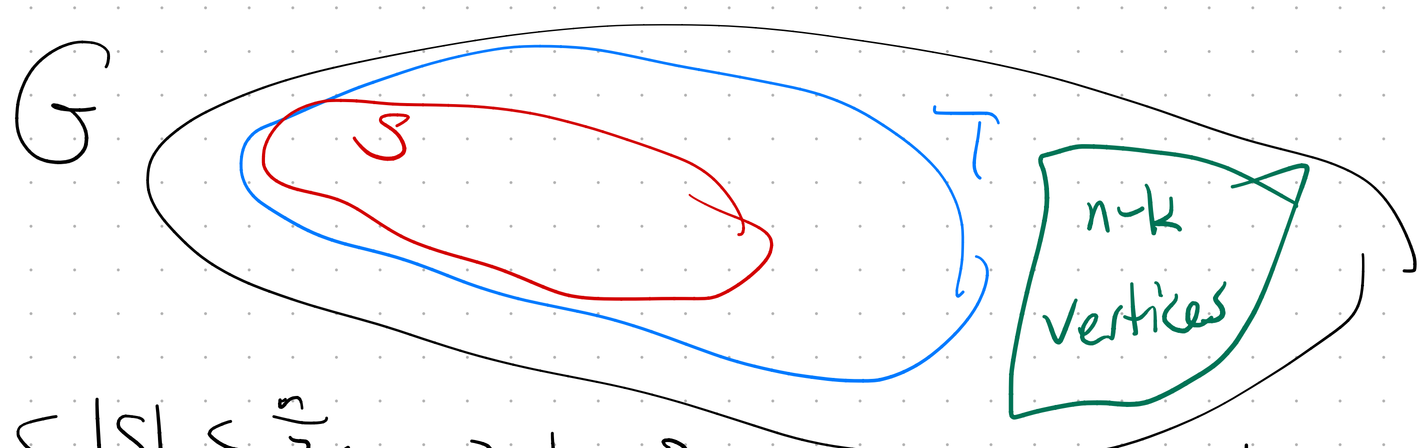
($\Uparrow$) If (S,T) is a $\frac{1}{2}$-ER then

$|S| \le \frac{n}{2}$ (2nd line of def.)

$\partial S \subseteq T \setminus S$ (3rd line of def)

$|\partial S| < \frac{1}{2}|S|$ (2nd line of def)

$T\setminus S$ has $< \frac{1}{2}$ as many elements as S.



$\frac{2}{3}k < |S| \le \frac{n}{2} \implies k < \frac{3}{4}n \implies n-k > \frac{1}{4}n.$

$\mathbb{E}[\#\text{ of } \tfrac{1}{2}\text{-ER in } G(n,p)]$

$$= \sum_{T} \sum_{S \subseteq T} \Pr(S,T \text{ is a } \tfrac{1}{2}\text{-ER})$$

$$\le \sum_{k=1}^{n} \binom{n}{k} \cdot 2^k \cdot (1-p)^{\left(\frac{2}{3}k\right)\left(\frac{1}{4}n\right)}$$

$$= \sum_{k=1}^{n} \binom{n}{k} \left[2(1-p)^{n/6}\right]^k$$

$$= -1 + \left(1 + 2(1-p)^{n/6}\right)^n$$

$$\leq -1 + \left[1 + 2e^{-pn/6}\right]^n$$

$p \geq \frac{7 \ln n}{n}$

$\frac{pn}{6} \geq \frac{7}{6} \ln(n)$

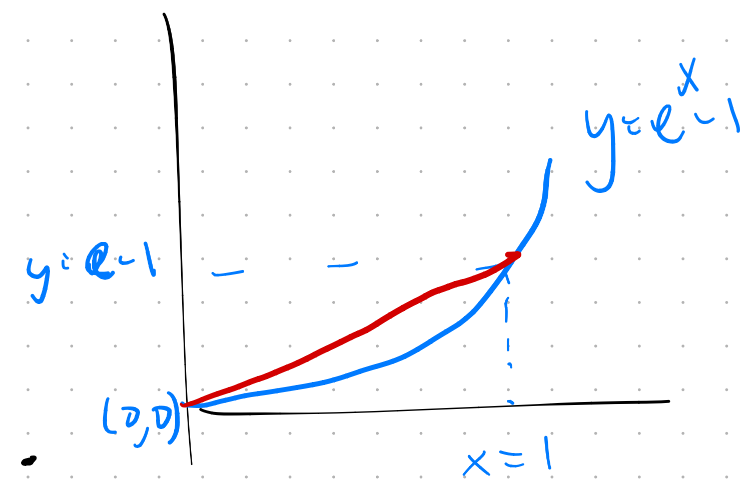
$$\leq -1 + \left(1 + 2n^{-7/6}\right)^n$$

$$\leq -1 + \left(e^{2n^{-7/6}}\right)^n$$

$$= -1 + e^{2n^{-1/6}}$$

[$n \geq 64$]

$$\leq (e-1) \cdot \frac{2}{n^{1/6}}.$$



By Markov Ineq,

$$\Pr\left(G(n,p) \text{ has a } \tfrac{1}{2}\text{-ER}\right) \leq \frac{2(e-1)}{n^{1/6}}$$

$$\Pr\left(G(n,p) \text{ is a } \tfrac{1}{2}\text{-expander}\right)$$

$$\geq 1 - \frac{2(e-1)}{n^{1/6}}.$$

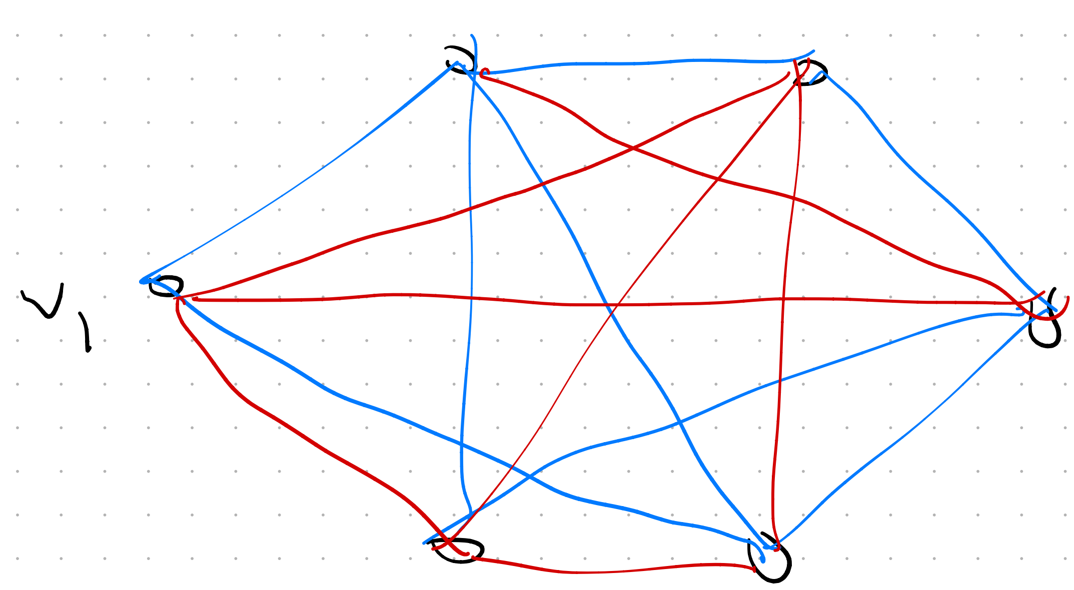
RAMSEY THEORY

Def. A clique in an undirected graph is a vertex set S such that every two elements of S form an edge.

An indep set in an undirected graph is a vertex set S such that every two elements of S don't form an edge.

Theorem. Every graph with ≥ 6 vertices contains either a clique or an independent set of size 3.

Proof. Take any 6 vertices $v_1, v_2, v_3, v_4, v_5, v_6$.



Connect every pair with a blue edge if $(v_i, v_j) \in E(G)$ red edge otherwise.

v_1 either has 3 blue neighbors or 3 red neighbors.

Assume red. If any 2 of those are connected by a red edge, we have a red triangle.

If they are all connected by blue edges, we have a blue triangle.

$R(3,3) \leq 6$.

$R(k,l)$ is the minimum n such that every n-vertex graph has either a clique of size k or indep set of size l.

Theorem [Ramsey, 1930]

$$R(k,l) \leq 2^{k+l-3},$$

Proof. Suppose $n = 2^{k+l-3}$.

Color every pair of vertices

blue if (v_i, v_j) is edge of G

red if not.

We seek a clique with all edges same color, either blue with k vertices or red with l vertices.

$t = k+l-3$.

Construct sequence of vertices

$v_0, v_1, \ldots, v_t$

and vertex sets $S_0 \supset S_1 \supset \ldots \supset S_t$.

Starting with v_0 = any vertex

S_0 = all vertices.

Always $v_i \in S_i$.

Look at colors of edges

from v_i to $S_i \setminus \{v_i\}$.

Majority red $\Longrightarrow$ S_{i+1} = red neighbors

Majority blue $\Longrightarrow$ S_{i+1} = blue neighbors

v_{i+1} = arbitrary element of S_{i+1}.

$$|S_{i+1}| \geq \left\lceil \frac{|S_i|-1}{2} \right\rceil.$$

$|S_0| \geq 2^{k+l-3}$

$\Longrightarrow$ $S_1, \ldots, S_t$ all non-empty.

$v_0, \quad v_1, \quad v_2, \quad \ldots, \quad v_{t-1}, \quad v_t$

If $k-1$ of the vertices in $v_0, \ldots, v_{t-1}$ are blue $\Longrightarrow$ blue k-clique.

If $l-1$ of them are red $\Longrightarrow$ red l-clique.

If $\leq k-2$ blue, $\leq l-2$ red $\Longrightarrow$ $\leq k+l-4$ colored vertices in $v_0, \ldots, v_{t-1}$ $\lightning$.

Thm (Erdős, 1947)

$$R(k,k) \geq 2^{\frac{k+1}{2}} \quad \forall\, k \geq 3.$$

Proof (next time).