17 Mar 2025 $G(n,p)$: connectivity, diameter, expansion

Recap: When $p \leq \frac{\ln n}{2n-2}$, $G(n,p)$ probably (w. prob. $1-o(1)$) has isolated vertices. ($\therefore$ disconnected)

When $p > (1+\varepsilon)\frac{\ln(n)}{n}$ $G(n,p)$ probably has no isolated vertices

... but is it connected?

Def. A *disconnecting partition* of a graph G is a partition of $V(G)$ into sets A, B, both nonempty, such that G has no edges from A to B.

Fact. A graph is connected if and only if it has no disconnecting partition.

Strategy for verifying $G(n,p)$ connected with high probability

1. Calculate $E[\#$ disconn partitions$]$
2. For p large enough, attempt to prove this expected value is $\ll 1$.

By linearity of expectation

$$E[\#\text{ disconn part'n}] = \sum_{\substack{\text{partitions } A,B \\ A\neq\emptyset,\ B\neq\emptyset}} \Pr(A,B \text{ is a disconn part'n})$$

Consider partition A, B with $|A|=k$, $|B|=n-k$

WLOG $k \le \frac{n}{2}$.

$$\Pr(A,B \text{ is a discon part'n}) = (1-p)^{k(n-k)}$$

$$\mathbb{E}[\#\text{ discon part'n}] \le \sum_{k=1}^{n/2} \binom{n}{k}(1-p)^{k(n-k)}$$

$$< \sum_{k=1}^{n/2} \binom{n}{k}(1-p)^{kn/2}$$

$$< -1 + \sum_{k=0}^{n} \binom{n}{k}\left[(1-p)^{n/2}\right]^k$$

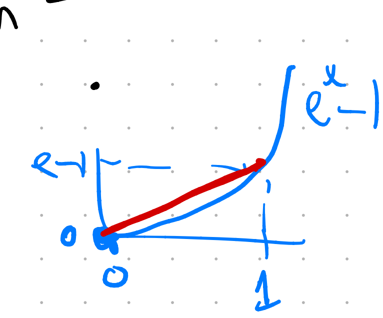
$$= -1 + \left[1 + (1-p)^{n/2}\right]^n$$

Recall $1-p < e^{-p}$ so $(1-p)^{n/2} < e^{-pn/2}$

Set $p \ge \frac{3\ln(n)}{n}$, then $(1-p)^{n/2} < \exp\left(-\frac{3\ln n}{\cancel{n}}\cdot\frac{\cancel{n}}{2}\right)$

$$= e^{-\frac{3}{2}\ln(n)} = n^{-3/2}$$

$$\left(1+n^{-3/2}\right)^n < \left(e^{n^{-3/2}}\right)^n = e^{n^{-3/2}\cdot n} = e^{n^{-1/2}}.$$



$$\Pr(G(n,p) \text{ disconnected}) < e^{n^{-1/2}} - 1$$

$$\le (e-1)\cdot n^{-1/2}.$$

Use $e^x-1 < (e-1)x$ when $0<x<1$.

The connectivity threshold for $G(n,p)$ is actually at $p \approx \frac{\ln(n)}{n}$, though we won't prove it here.

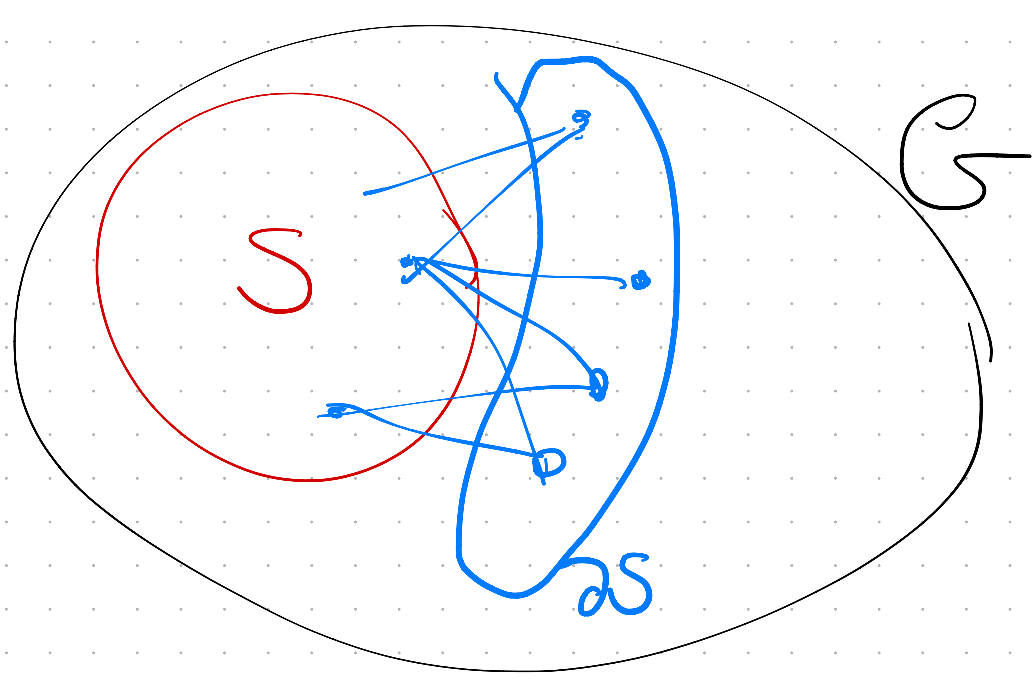
<u>Def.</u> Graph G with n vertices is called an <u>α-expander</u> if $\forall$ vertex set S with $|S| \leq \frac{n}{2}$, the set

scalar value

"vertex boundary of S"

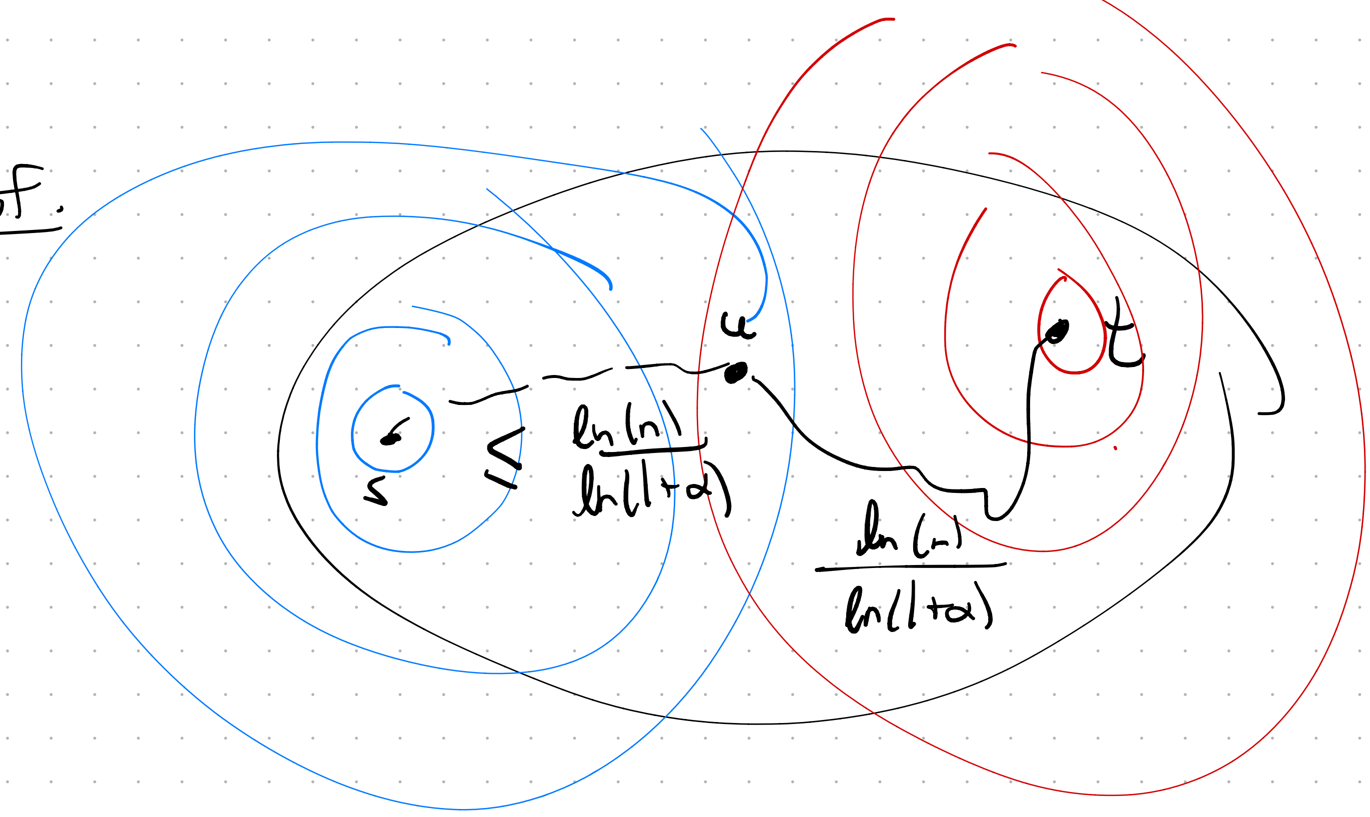
$$\boxed{\partial S} = \{ v \mid v \notin S \text{ but } \exists u \in S \text{ s.t. } (v,u) \in E(G) \}$$

has at least $\alpha |S|$ elements.



<u>Lemma.</u> If G ($\forall$ graphs G) is an α-expander with n vertices then the diameter of G is $\leq 2 \frac{\ln(n)}{\ln(1+\alpha)}$.

Proof.



α- expansion $\Longrightarrow$ ball of radius r around $\{s\}$ has $\geq (1+\alpha)^r$ vertices unless it has $> n/2$ vertices.

... and same for t.

Let $B_r(s) = \{$vertices reachable from s by a path of r or fewer edges$\}$.

$\partial B_r(s) = \{$vertices not reachable from s in r "hops" but they are one hop away from such vertex$\}$

$= \{$vertices at distance exactly $r+1$ from $s\}$

$= B_{r+1}(s) \setminus B_r(s)$

α-Expansion $\Rightarrow$ $|\partial B_r(s)| \geq \alpha \cdot |B_r(s)|$ if $|B_r(s)| \leq \frac{n}{2}$.

$\Longrightarrow |B_{r+1}(s)| \geq (1+\alpha) \cdot |B_r(s)|$ if $|B_r(s)| \leq \frac{n}{2}$.

Induction starting from $|B_0(s)| = 1 = (1+\alpha)^0$.

$\Longrightarrow |B_{r+1}(s)| \geq (1+\alpha)^{r+1}$ if $|B_r(s)| \leq \frac{n}{2}$.

When $r+1 = \left\lceil \frac{\ln n}{\ln(1+\alpha)} \right\rceil$

$$(1+\alpha)^{r+1} \geq n.$$

Possibilities -

$$|B_{r+1}(s)| \geq (1+\alpha)^{r+1} \geq n$$

$\Longrightarrow$ all vertices reachable in $r+1$ hops from s.

$$|B_{r+1}(t)| \geq (1+\alpha)^{r+1} \geq n$$

$\Longrightarrow$ all vertices reachable in $r+1$ hops from t.

$$|B_{r+1}(s)|, \; |B_{r+1}(t)| < (1+\alpha)^{r+1}$$

$\Longrightarrow$ $|B_r(s)| > \frac{n}{2}$, $|B_r(t)| > \frac{n}{2}$

$\Longrightarrow$ $B_r(s)$, $B_r(t)$ cannot be disjoint

$\Longrightarrow$ $\exists$ a vertex u that can reach both s & t in $\leq r$ hops.

All cases $\Longrightarrow$ diameter$(G) \leq 2r$.

"Region growing argument"