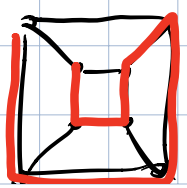
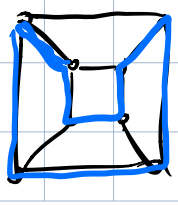
5 April 2021 Hamiltonian Cycle

Announcement. Prelim 2 is April 13 – 15.
See Ed post "Prelim 2 Time Slot Survey"
Fill out survey by 23:59 on Tuesday.
If you need accommodations and didn't have Prelim 1 accommodations, ask SDS to notify Prof. Kleinberg.
Subject matter: Chapters 5, 7, 8
Div. & Conquer
Network Flow (Problem Sets 5, 6, 7.)
NP-Completeness
Drop deadline is today, April 5.

Def. A Hamiltonian path in a graph is a path that visits every vertex exactly once. A Hamiltonian cycle is a Ham. path that has an edge from its ending point to its starting point.





One can ask these questions about directed or undirected graphs.

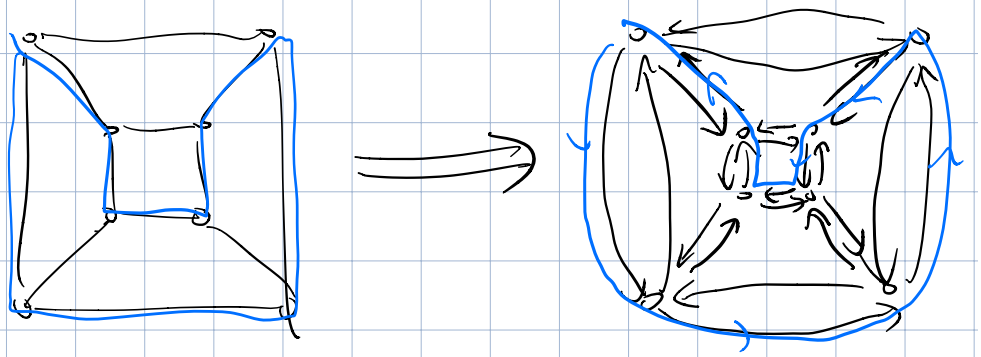
In all we have 4 problems:

Dir Ham Cycle — Dir Ham Path

Undir Ham Cycle — Undir Ham Path

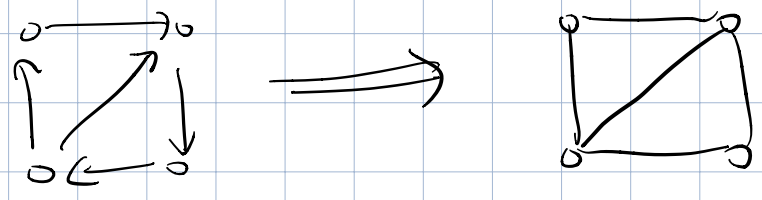
These are all computationally equivalent.

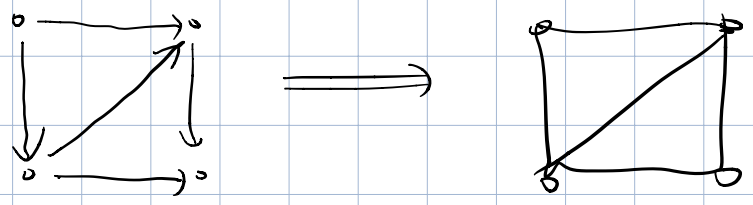
Reducing UNDIR HAM CYCLE to DIR HAM CYCLE.



Reducing DIR HAM CYCLE to UNDIR HAM CYCLE.

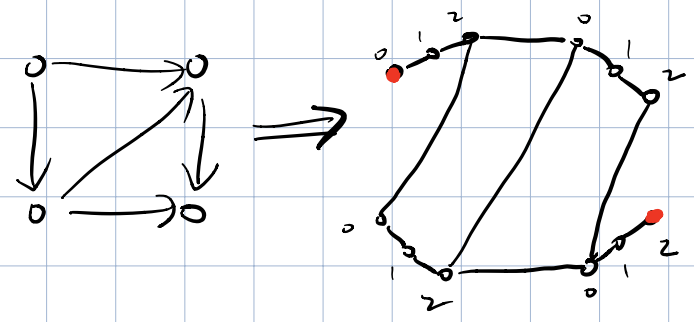
first idea:

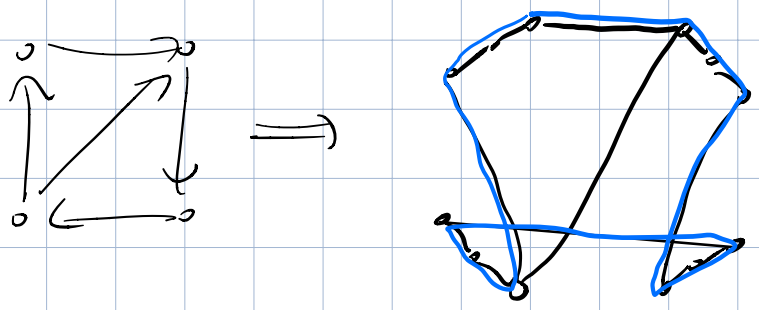




Second idea: Turn every vertex of the directed graph, v, into 3 vertices v_0, v_1, v_2 joined into a path.

Turn directed edge (v, w) into undirected edge (v_2, w_0).





Why this works: if dir graph has a Ham cycle, the undir

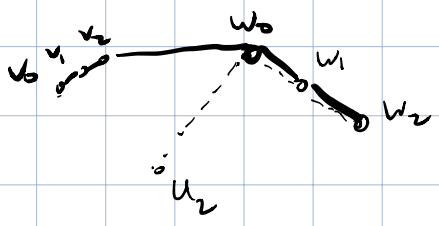
graph has one as well.
(Go through same sequence of vertices, visiting their copies in the order 0, 1, 2.)

Conversely, if the undir graph has a Ham cycle, take some vertex v in the dir graph, with corresponding triad v_0, v_1, v_2 in the undir graph. The Ham cycle must visit v_1. Immediately before and after, it must visit v_0 and v_2. (They are the only 2 neighbors of v_1.)

Assume WLOG the order is v_0, v_1, v_2. (Otherwise reverse the cycle.)

A type-2 vertex such as v_2 has:
- one neighbor of type 1, v_1
- some neighbors of type 0
- no neighbors of type 2

So next after v_2 is w_0 for some w.



After w_1 it must visit w_1, w_2 or else w_1 becomes "stranded."

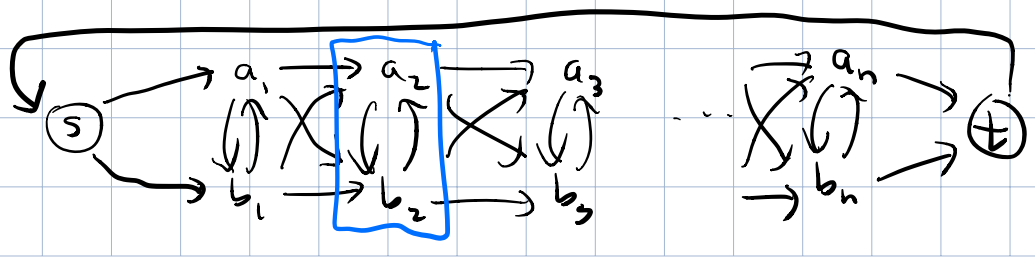
Continue this reasoning inductively: the undirected Ham cycle visits all 3 nodes of each "vertex gadget" consecutively in the order 0, 1, 2, and hence it can be "decoded" to a directed Ham cycle by collapsing each triad (v_0, v_1, v_2) into a single vertex of the directed graph.

3SAT $\leq_p$ DIRECTED HAM CYCLE

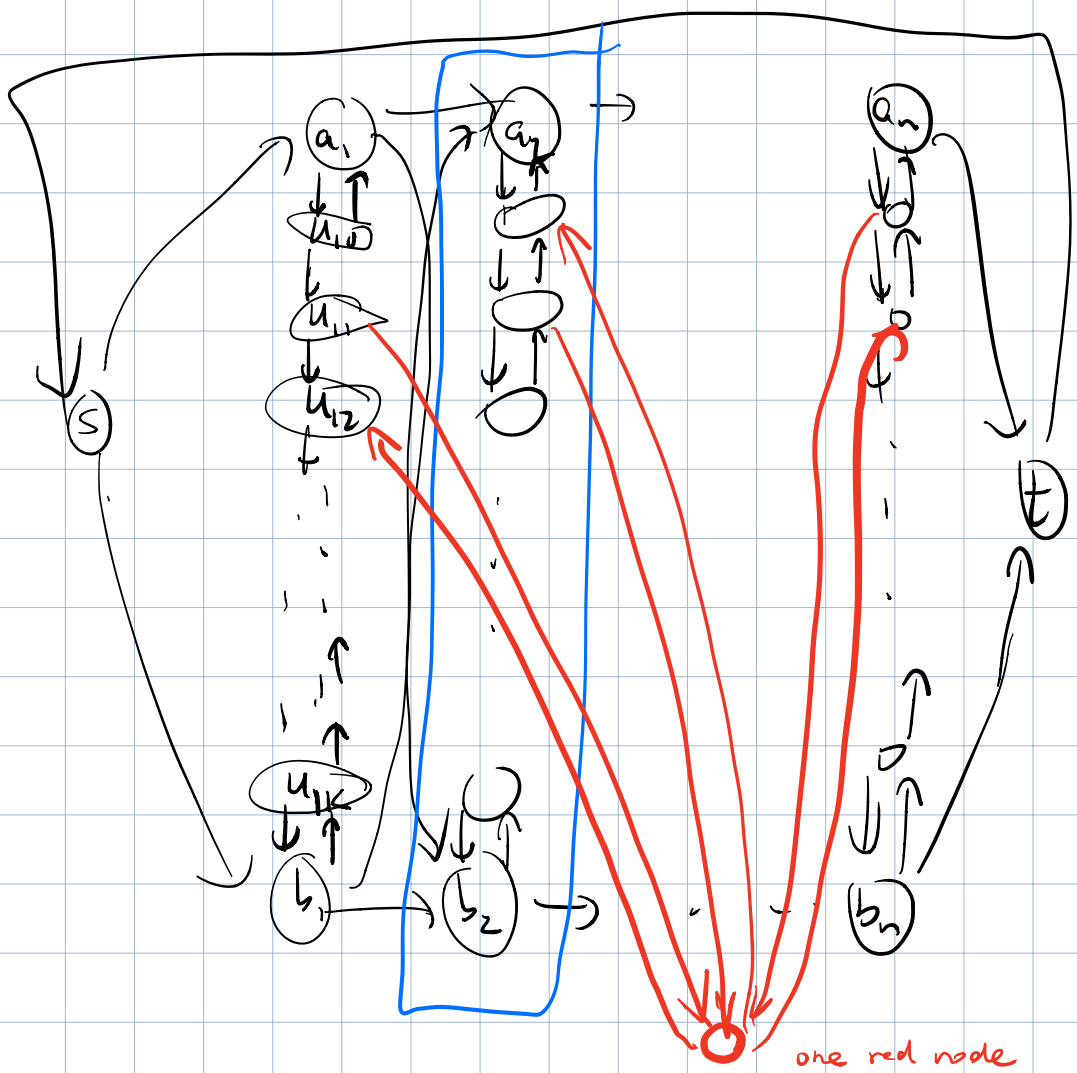
Reducing 3SAT to another problem

STEP 1. Find a gadget that forces a binary (yes/no) decision. Make n of these gadgets.

STEP 2. Find a gadget that "enforces a 3-strikes-and-out" by penalizing a combination of 3 wrong decisions.



This graph has 2^n Hamiltonian cycles corresponding to the 2^n truth assignments of a 3SAT formula.



clause $a_1 \vee \overline{a_2} \vee a_n$