19 Mar 2021

Finishing Push–Relabel
Bipartite Max Matching

Excess of v: $\quad x(v) = \sum_{e \text{ into } v} f(e) - \sum_{e \text{ out of } v} f(e)$

Steepness condition: if $(v,w) \in E(G_f)$ $\quad h(v) \leq h(w) + 1$.

Residual capacity $\quad c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \text{ forward} \\ f(v,u) & \text{if } (u,v) \text{ backward} \end{cases}$

PUSH–RELABEL.

initialize $\quad f(e) = \begin{cases} c(e) & \text{if } e \text{ is edge out of } s \\ \phi & \text{otherwise} \end{cases}$

initialize $\quad h(v) = \begin{cases} n & \text{if } v = s \\ \phi & \text{otherwise.} \end{cases}$

Compute $c_f(e)$ $\quad \forall\, e \in E(G_f)$

Compute $x(v)$ $\quad \forall\, v$

while $\exists u \neq t$ with $x(u) > 0$:

  if G_f contains an edge (v,w) such that $h(v) > h(w)$ and $x(v) > 0$:

    PUSH(v,w): let $\delta = \min\{x(v), c_f(v,w)\}$

      if (v,w) is forward edge

        $f(v,w) \leftarrow f(v,w) + \delta$

      if (v,w) is backward edge

        $f(w,v) \leftarrow f(w,v) - \delta$

update $c_f(v,w)$, $c_f(w,v)$, $x(v)$, $x(w)$

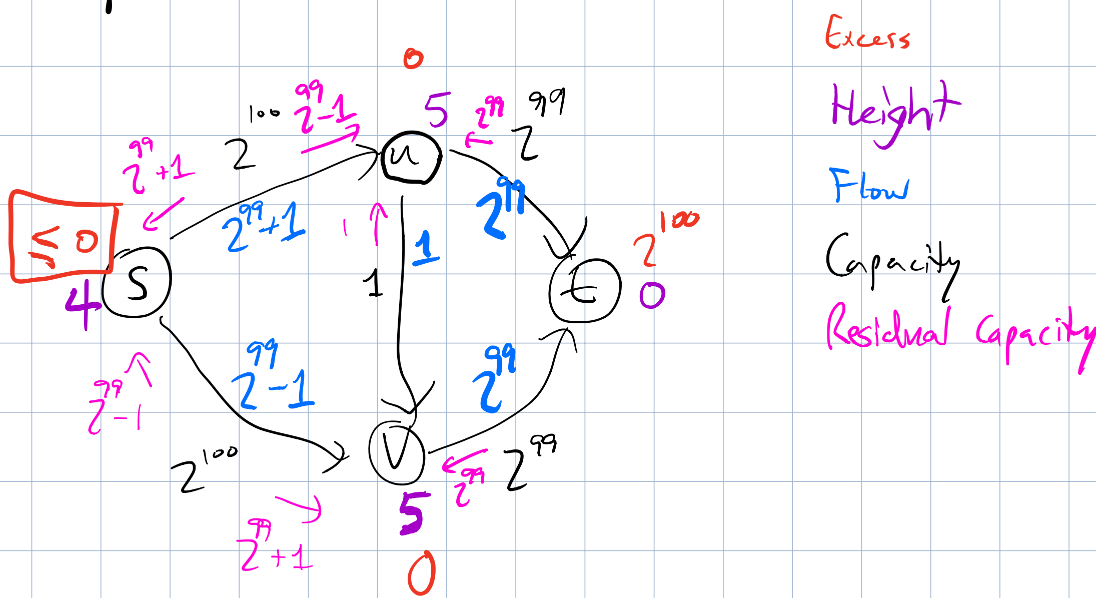
else:

find v = maximum height vertex in $\{u \neq t \mid x(u) > t\}$

RELABEL(v): $h(v) \leftarrow h(v) + 1$.

endwhile

output f



<u>Analysis:</u>

<u>Obs 1.</u> If the alg terminates, f is a flow.

<u>Obs. 2.</u> If $x(v) > 0$ then G_f contains a path from v to s.

Obs. 3. $h(s) = n$ and $h(t) = 0$ throughout the algorithm.

Reason. $x(s) \leq 0$ so never relabel it. Code never allows relabeling sink.

Obs. 4. G_f never contains a path from s to t.

Reason. $h(s) = n$, $h(t) = 0$, each edge of G_f reduces height by ≤ 1. (steepness condition.) So a path from s to t must have $\geq n$ edges, $\geq n+1$ vertices. The graph has only n vertices.

Proposition: The push-relabel algorithm computes a max flow, if it terminates.

Proof. We saw that when it term's, f is a flow and G_f has no s-t path. When we proved correctness of Ford-Fulkerson we saw these 2 properties guarantee max flow.

Bounding running time.

Obs. 5. $\forall v \quad h(v) \le 2n$ at all times.

Proof. When we RELABEL(v) to increase height, $x(v) > 0$.

$\Longrightarrow$ G_f contains $v \longrightarrow s$ path.

$\Longrightarrow$ height diff $h(v) - h(s) < n$. [steepness condition]

$\Longrightarrow$ $h(v) \le 2n-1$ before RELABEL.

Obs. 6. At most $2n^2$ RELABEL ops.

To bound #pushes, we distinguish two types.

SATURATING: use up all residual cap of (v,w) and remove it from G_f.

We won't do another saturating (v,w) push until PUSH(w,v) which puts (v,w) back into G_f.

RELABEL(w) must happen twice

before PUSH(w,v).

$\Longrightarrow$ # saturating PUSH$(v,w) \leq n$.

In total $O(mn)$ saturating pushes.
$\leq O(n^3)$.

NONSATURATING: also $O(n^3)$.
See book.

$\boxed{\text{There exists a } O(n^3) \text{ algorithm for solving max-flow.}}$