$\boxed{\text{22 Feb 2021}}$ Weighted interval scheduling (§6.1)

Announcement.

Problem Set 2, Question 2.

You can assume log entries have distinct timestamps. However u_i's, v_i's (or even u_i, v_i pairs) can occur repeatedly.

Also don't assume log entries are in any particular order.

DYNAMIC PROGRAMMING (Chapter 6)

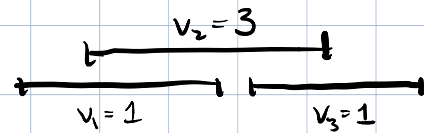
Weighted interval scheduling.

Input consists of n requests or jobs.

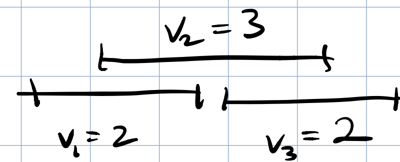
Each has three properties:

- start time s_i
- finish time f_i
- value $v_i \geq 0$ if selected.

GOAL: Select a set of jobs that is conflict-free (if i, j are selected then $[s_i, f_i]$ is disjoint from $[s_j, f_j]$) and has max total value subject to conflict-free.

EFT no longer works: 

optimal set is job 2 only.
EFT picks jobs 1 and 3.

Greedy by value also doesn't work: 

optimal set is jobs 1 and 3.
Greedy by value gets only job 2.

In fact nobody knows a greedy algorithm that solves every instance of WIS correctly. There probably isn't one, but it depends on how you define a greedy algorithm, which is still not standardized.

Plan: Think more carefully and systematically about the set of feasible solutions and how to optimize over it.

Structure Lemma for Conflict-Free Request Sets

Assume requests are sorted such that $f_1 \le f_2 \le \cdots \le f_n$. Let $p(n)$ denote the highest numbered request that finishes before s_n.

Every conflict-free subset of $\{1,2,\dots,n\}$ is either:

(a) a conflict-free subset of $\{1,2,\dots,n-1\}$

(b) $S \cup \{n\}$ where S is a conflict free subset of $\{1,\dots,p(n)\}$.

Proof sketch. The lemma statement encodes the obvious fact that set of requests either contains n or it doesn't. And if it contains n and is conflict-free than it can't contain any interval whose finish time is in $[s_n, f_n]$.

Recursive algorithm for computing the maximum value of a conflict free set. (Not the contents of the set.)

Compute-Opt(n): // Compute max value of a conflict free subset of $\{1,\dots,n\}$.

if $n=0$ return 0

else

find $p(n)$.

return $\max\{\underbrace{\text{Compute-Opt}(n-1)}_{\text{case (a)}}, \underbrace{v_n + \text{Compute-Opt}(p(n))}_{\text{case (b)}}\}$

Correctness of Compute-Opt(n) follows by induction on n. Ind hyp is that the function correctly computes the max value of a conflict free subset of $\{1, 2, \dots, n\}$.

Base case $n = 0$: the only subset of the empty set is empty and has value $\emptyset$.

Induction step: Structure lemma implies that max-value conflict free subset is either

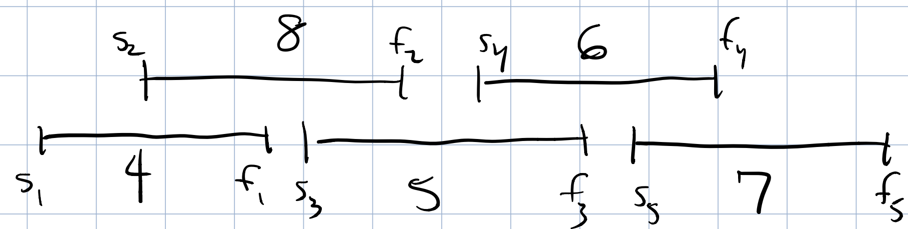
(a) the max value conf-free subset of $\{1, \dots, n-1\}$; or

(b) $S \cup \{n\}$ where S is the max value conf-free subset of $\{1, \dots, p(n)\}$.

By induct hyp, max value achievable in case (a) is Compute-Opt(n-1).

And max value achievable in case (b) is v_n + Compute-Opt(p(n)).

$\therefore$ Compute-Opt(n) outputs the right answer.

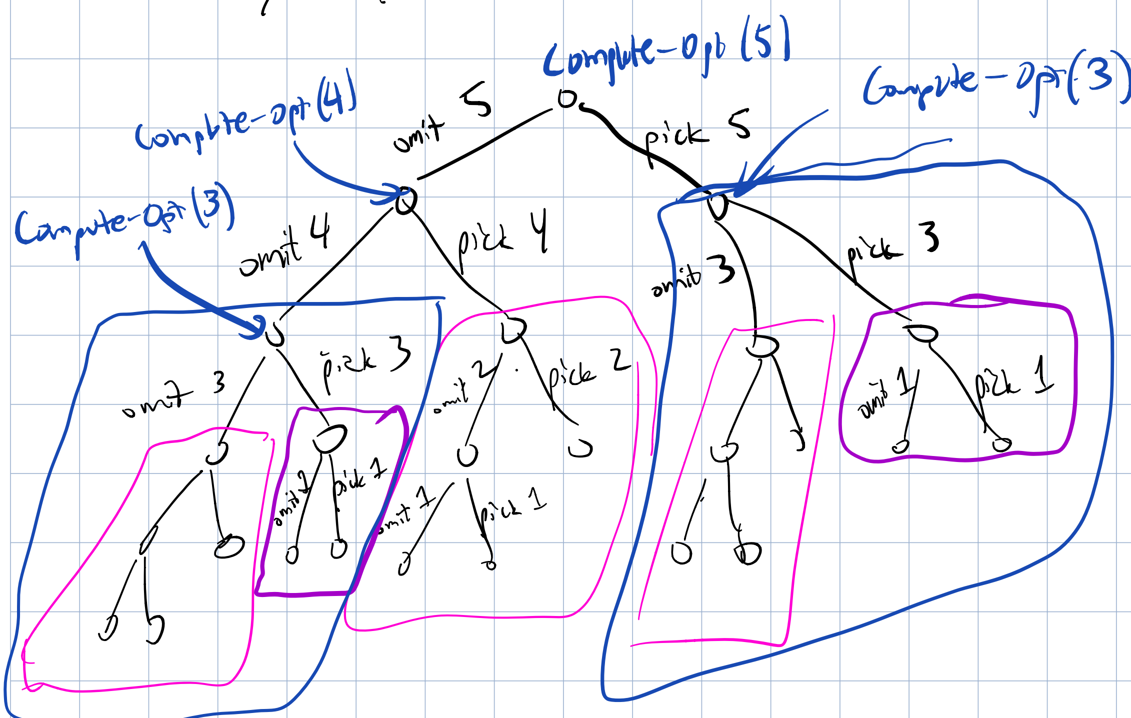


Struct Lemma says: A conf-free subset either picks $[s_5, f_5]$ ~~together~~ with a subset of first 3 requests, or it omits $[s_5, f_5]$.

Case a

Case b

If you pick $[s_5, f_5]$ you get 7 plus the value of the other jobs picked. Otherwise you get the value of whatever subset of the first 4 jobs you pick.



Running the pseudocode in blue above would generate a set of recursive calls to compute-opt(...) modeled by the tree above.

Watch out! The tree can get exponentially big. To make the algorithm efficient we'll use "memoization." Store results of previous Compute-Opt calls in a table, M[].

Compute-Opt(n):
if M[n] is non-null, return M[n].
else // This is the first time we've been asked to solve Compute-Opt(n).

if $n=0$
set $M[n]=0$
else
compute $p(n) = \max\{i \mid f_i < s_n\}$
set
$$M[n] = \max\left\{\begin{array}{l} \text{Compute-Opt}(n-1), \\ v_n + \text{Compute-Opt}(p(n)) \end{array}\right\}$$

return M[n].

Analysis of running time:

Excluding time spent in recursive calls Compute-Opt does $O(n)$ work. (Computing $p(n)$.)

Furthermore, Compute-Opt (i) does this work at most once, for each $i = 0, 1, 2, \ldots, n$.

Running time, in total is

$$\sum_{i=0}^{n} (\text{time spent on Compute-Opt}(i))$$

$$\leq n \cdot O(n)$$

$$= O(n^2).$$

Faster implementation: pre-compute $p(i)$ for each i. This preprocessing step happens before we ever call Compute-Opt.

The preprocessing to compute $p(1), p(2), \ldots, p(n)$ can be implemented to run in $O(n)$.

Then Compute-Opt does $O(1)$ work outside recursive calls, so the whole algorithm becomes $O(n)$.