Introduction to Algorithms	**Problem Set 5**
CS 482, Spring 2008	**Due Friday, March 7**

(1) Let G be any flow network with integer edge capacities.

(a) Prove that for every edge e of G, the following two statements are equivalent.

1. There exists a minimum-capacity cut containing e.
2. In every maximum flow, edge e is saturated, i.e. $f(e) = c_e$.

In other words, prove that statement (1) implies statement (2) and vice-versa.

(b) Prove that there exists a maximum flow f in G, such that the set of edges carrying positive flow is acyclic. In other words, f has the property that for every directed cycle in G made up of edges $e_1, e_2, \ldots, e_k$, at least one of the values $f(e_1), f(e_2), \ldots, f(e_k)$ is equal to zero.

(2) Solve Chapter 7, Exercise 11 in Kleinberg & Tardos.

(3) We are given an $n \times n$ matrix $A = (a_{ij})$ with non-negative entries, and we are interested in the question of whether it can be expressed as a sum of two matrices R, C such that:

1. The entries of R and C are non-negative.
2. The row sums of R are bounded above by 1.
3. The column sums of C are bounded above by 1.

When there exist matrices R, C satisfying these three constraints, such that $R + C = A$, let us call the pair (R, C) a *row+column decomposition of A.*

(a) Prove that A has a row+column decomposition **if and only if** for every set of rows S and columns T, we have

$$\sum_{i \in S, j \in T} a_{ij} \leq |S| + |T|.$$

(b) Design an efficient algorithm which takes a non-negative matrix A as input, and either outputs a row+column decomposition or reports (correctly) that no such decomposition exists.