Automata and Computability	**Proofs of (9.14)–(9.18), p. 50**
CS481 Fall 2002	**Monday, September 23, 2002**

Here are proofs of (9.14)–(9.18), p. 50, as requested. I'll use the notation of Supplementary Lecture A and refer to the axioms (A.1)–(A.13) instead of (9.1)–(9.13). First let's establish some useful lemmas.

Lemma 1 The operations $+$, $\cdot$, and * are *monotone* with respect to $\leq$; that is, for all a, b, c, if $a \leq b$, then

(i) $a + c \leq b + c$,

(ii) $c + a \leq c + b$,

(iii) $ac \leq bc$,

(iv) $ca \leq cb$,

(v) $a^* \leq b^*$.

To show (i),

$$\begin{array}{rcll} a \leq b & \Rightarrow & a + b = b & \text{definition of } \leq \\ & \Rightarrow & a + b + c = b + c & \\ & \Rightarrow & a + c + b + c = b + c & \text{idempotence, commutativity of } + \\ & \Rightarrow & a + c \leq b + c & \text{definition of } \leq. \end{array}$$

Property (ii) follows from (i) and commutativity of $+$. To show (iii),

$$\begin{array}{rcll} a \leq b & \Rightarrow & a + b = b & \text{definition of } \leq \\ & \Rightarrow & (a + b)c = bc & \\ & \Rightarrow & ac + bc = bc & \text{distributivity} \\ & \Rightarrow & ac \leq bc & \text{definition of } \leq. \end{array}$$

The argument for (iv) is symmetric. For (v), we have $1 + ab^* \leq 1 + bb^* = b^*$ by monotonicity and (A.10). Then $a^* \leq b^*$ follows from (A.12).

Lemma 2 $a^*a^* = a^*$.

To show the inequality $\leq$, by (A.12) it suffices to show $a^* + aa^* \leq a^*$.

$$\begin{array}{rcll} a^* + aa^* & = & 1 + aa^* + aa^* & \text{by (A.10)} \\ & = & 1 + aa^* & \text{by idempotence} \\ & = & a^* & \text{by (A.10).} \end{array}$$

To show the reverse inequality, by (A.12) it suffices to show $1 + aa^*a^* \leq a^*a^*$.

$$\begin{array}{rcll} 1 + aa^*a^* & \leq & 1 + aa^* + aa^*a^* & \\ & = & a^* + aa^*a^* & \text{(A.10)} \\ & = & (1 + aa^*)a^* & \text{distributivity} \\ & = & a^*a^* & \text{(A.10).} \end{array}$$

Lemma 3 $\quad a^{**} = a^*$.

To show $\leq$, by (A.12) it suffices to show $1 + a^*a^* \leq a^*$. We have $1 \leq a^*$ by (A.10) and $a^*a^* \leq a^*$ by Lemma 2, thus $a^{**} \leq a^*$ since $+$ gives the least upper bound with respect to $\leq$.

For the reverse inequality, we know $a \leq a^*$, since by (A.10) and distributivity, $a^* = 1 + aa^* = 1 + a(1 + aa^*) = 1 + a + aaa^*$. Then $a^* \leq a^{**}$ follows from monotonicity of *.

(9.14) I did this one in class.

(9.15) To prove $(a^*b)^*a^* = (a + b)^*$, it suffices to prove inequalities in both directions.

To show $(a^*b)^*a^* \leq (a + b)^*$,

$$\begin{array}{rcll} (a^*b)^*a^* & \leq & ((a + b)^*(a + b))^*(a + b)^* & \text{monotonicity} \\ & \leq & (a + b)^{**}(a + b)^* & \text{(A.11) and monotonicity} \\ & = & (a + b)^*(a + b)^* & \text{Lemma 3} \\ & = & (a + b)^* & \text{Lemma 2.} \end{array}$$

For the reverse inequality, by (A.12) it suffices to show $1 + (a + b)(a^*b)^*a^* \leq (a^*b)^*a^*$. By distributivity and the fact that $+$ gives the least upper bound with respect to $\leq$, it suffices to show

(a) $1 \leq (a^*b)^*a^*$
(b) $a(a^*b)^*a^* \leq (a^*b)^*a^*$
(c) $b(a^*b)^*a^* \leq (a^*b)^*a^*$.

The inequality (a) follows from two applications of (A.10) and monotonicity. For (b),

$$\begin{array}{rcll} a(a^*b)^*a^* & = & a(1 + a^*b(a^*b)^*)a^* & \text{(A.10)} \\ & = & aa^* + aa^*b(a^*b)^*a^* & \text{distributivity (A.8) and (A.9)} \\ & \leq & a^* + a^*b(a^*b)^*a^* & \text{(A.10) and monotonicity} \\ & = & (1 + a^*b(a^*b)^*)a^* & \text{distributivity (A.9)} \\ & = & (a^*b)^*a^* & \text{(A.10).} \end{array}$$

Finally, for (c), by (A.10) and monotonicity we have

$$b(a^*b)^*a^* \quad \leq \quad a^*b(a^*b)^*a^* \quad \leq \quad (a^*b)^*a^*.$$

(9.16) This follows immediately from (9.14) and (9.15).

(9.17) The inequality $\geq$ is immediate from monotonicity. For $\leq$, using distributivity, commutativity, (A.10), and idempotence, we have

$$\begin{aligned} 1 + (1+a)a^* &= 1 + a^* + aa^* \\ &= 1 + aa^* + a^* \\ &= a^* + a^* \\ &= a^*, \end{aligned}$$

therefore $(1+a)^* \leq a^*$ by (A.12).

(9.18) For $\leq$, by distributivity and (A.11) we have $a + a^*aa = (1 + a^*a)a = a^*a$, therefore $aa^* \leq a^*a$ by (A.13). The argument for the reverse inequality is symmetric.