Math 335 HW 6 - Due March 12, 2004

1. Let F be a field.

   (a) Let $f(x)$ be a polynomial of degree d in $F[x]$. Prove that f has at most d distinct roots in F. (A root of f is an element a such that $f(a) = 0$.)

   (b) Suppose that F is a finite field with p^k elements. Show that $x^{p^k} - x$ has exactly p^k distinct roots in F.

   (c) Suppose that F is a finite field with p^k elements. Let $f(x) \in F[x]$, where the degree of f is d and assume that $f(x)$ divides $x^{p^k} - x$. Show that f has d distinct roots in F.

2. Let F be a field. Let f be a degree d polynomial in $F[x]$ and let g be a degree d' polynomial in $F[x]$ with $d' < d$. Write the Euclidean algorithm as follows:

$$\begin{array}{ccc} f & = & q_0 g + r_1 \\ g & = & q_1 r_1 + r_2 \\ r_1 & = & q_2 r_2 + r_3 \\ \vdots & & \vdots + \vdots \\ r_{m-1} & = & q_m r_m + r_{m+1} \end{array}$$

   with $r_{m+1} = (f, g)$. Give the best estimate you can for m.

3. Problem 5.6 of the text.

4. Problem 5.7 of the text.

5. Problem 5.10 of the text.

6. Problem 5.11 of the text.

7. A group G is called *cyclic* if there is some $g \in G$ such that every element of G is a power of g. For each group below determine whether or not the group is cyclic.

   (a) Z_{11} with addition.

   (b) $(Z_{24})^\star$ with multiplication.

   (c) $(\mathbb{Z}_2[x]/(x^3 + x + 1))^\star$ with multiplication.

8. List all the right cosets of $\{0, 5, 10\}$ as a subgroup of $\mathbb{Z}_{15}$ (with addition). Also list the left cosets of $\{0, 5, 10\}$ as a subgroup of $\mathbb{Z}_{15}$.

   Recall that a permutation of $\{1, 2, 3\}$ is a bijection $\pi : \{1, 2, 3\} \to \{1, 2, 3\}$. Let G be the group of all permutations of $\{1, 2, 3\}$. (There are six of them.) Multiplication is composition of functions. For instance, if ϕ is the permutation which maps $1 \to 1, 2 \to 3, 3 \to 2$ and ψ is the permutation which maps $1 \to 2, 2 \to 3$ and $3 \to 1$, then $\pi_1 \cdot \pi_2$ is the permutation which maps $1 \to 3, 2 \to 2$ and $3 \to 1$. Let H be the subgroup of G consisting of two permutations π_1, which maps $1 \to 1, 2 \to 2$ and $3 \to 3$, and π_2 which maps $1 \to 2, 2 \to 1$ and $3 \to 3$. List the left cosets and the right cosets of H as a subgroup of G.