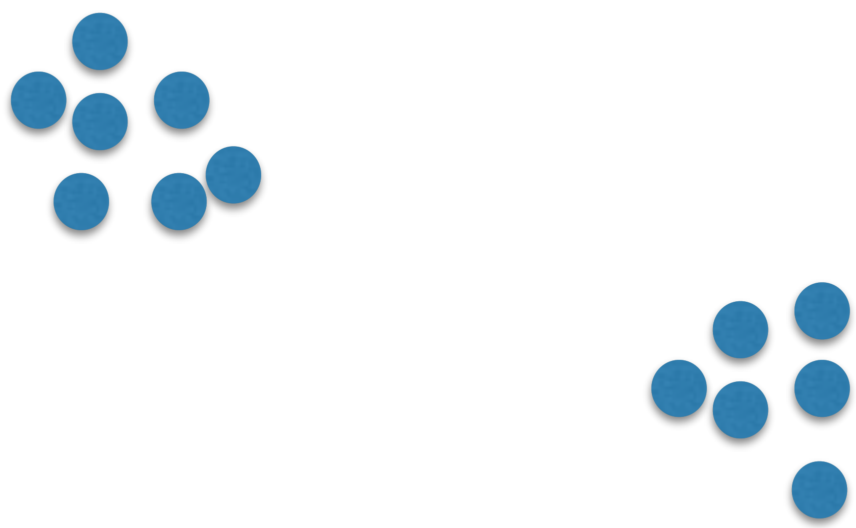
Machine Learning for Data Science (CS4786)
Lecture 11

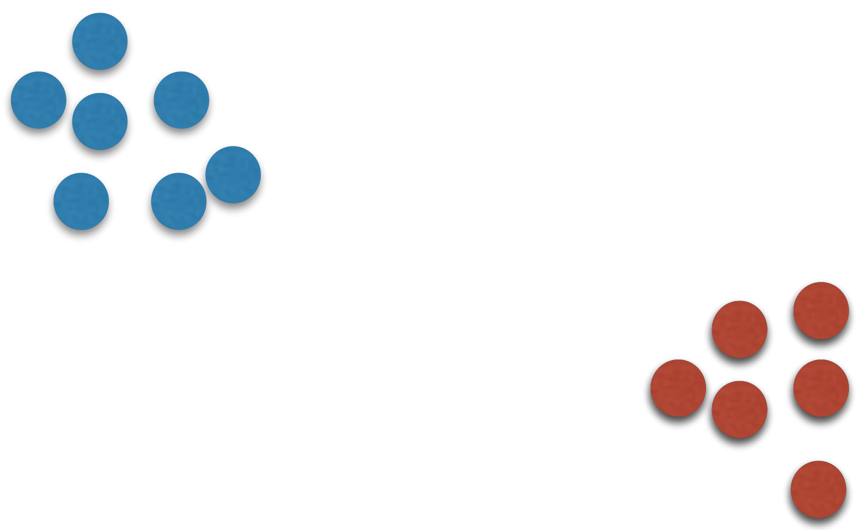
Clustering + Linkage Clustering

Examples



What are the clusters?

Examples



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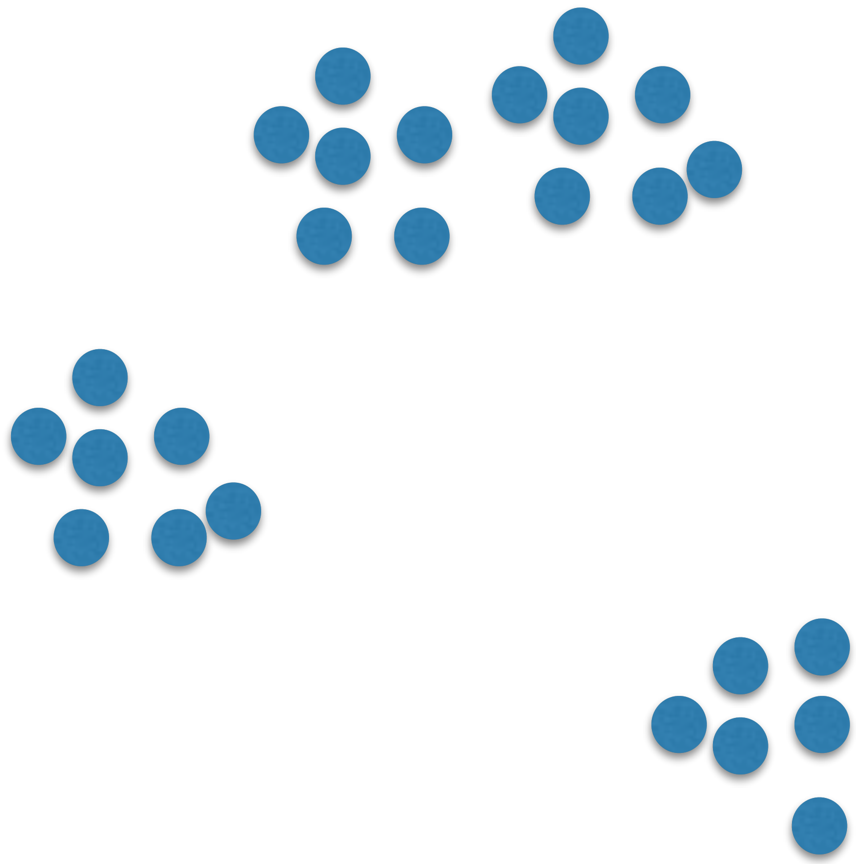

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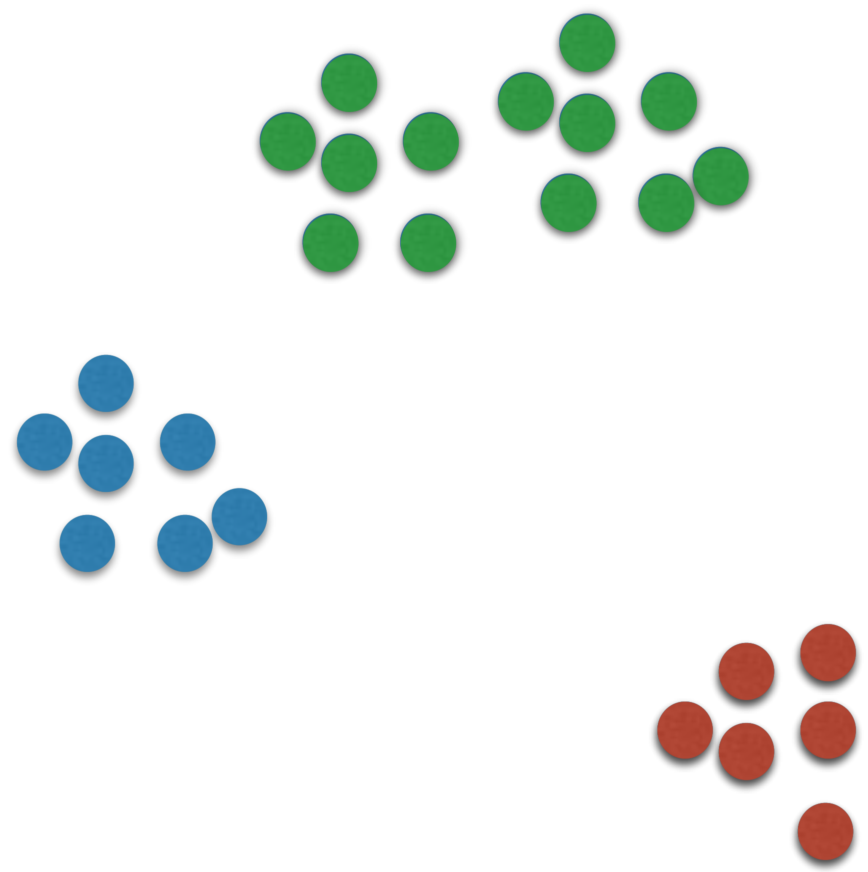

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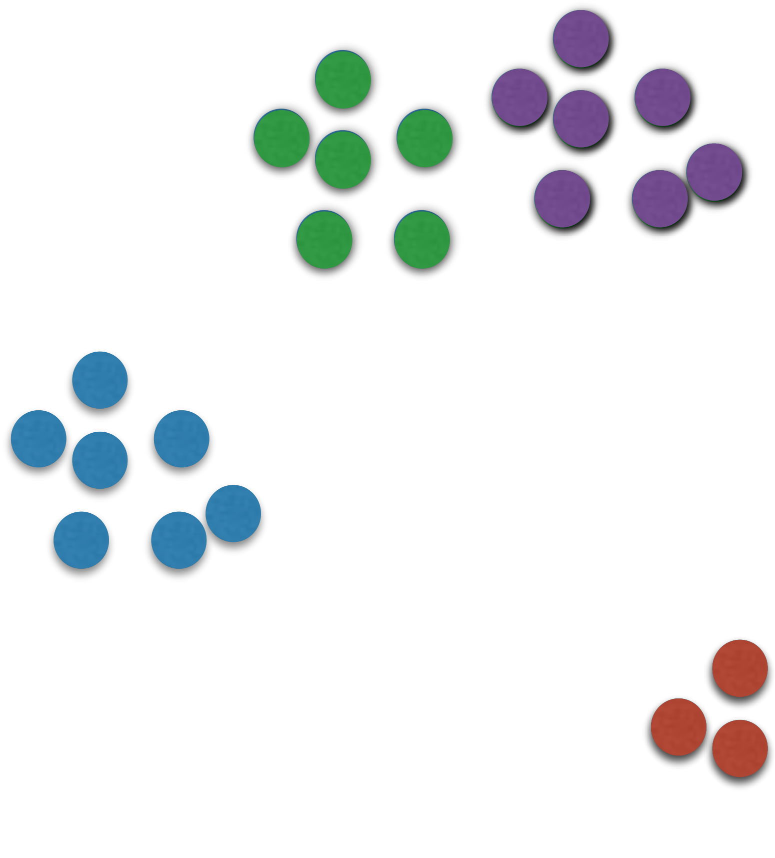

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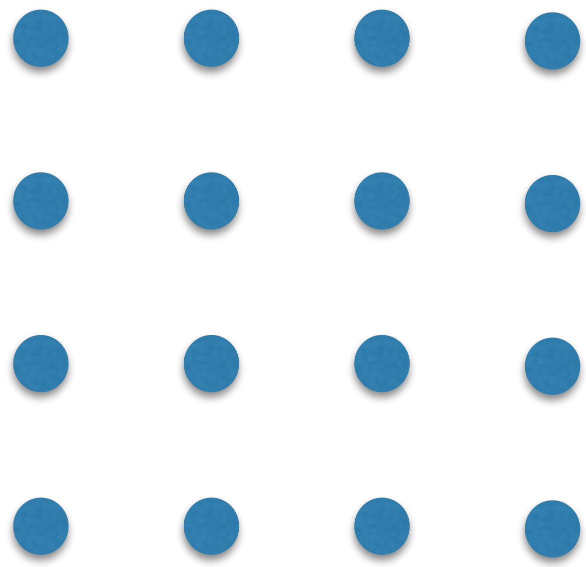
EXAMPLES



What are the clusters?

Clustering

- Grouping sets of data points s.t.
  - points in same group are similar
  - points in different groups are dissimilar
- A form of unsupervised classification where there are no predefined labels

Some Notations

- Kary clustering is a partition of $\mathbf{x}_1, \ldots, \mathbf{x}_n$ into K groups
- For now assume the magical K is given to use
- Clustering given by $C_1, \ldots, C_K$, the partition of data points.
- Given a clustering, we shall use $c(\mathbf{x}_t)$ to denote the cluster identity of point $\mathbf{x}_t$ according to the clustering.
- Let n_j denote $|C_j|$, clearly $\sum_{j=1}^{K} n_j = n$.

How do we formalize a good clustering objective?

How do we formalize?

Say dissimilarity$(\mathbf{x}_t, \mathbf{x}_s)$ measures dissimilarity between $\mathbf{x}_t$ & $\mathbf{x}_s$

Given two clustering $\{C_1, \ldots, C_K\}$ (or c) and $\{C'_1, \ldots, C'_K\}$ (or c')

How do we decide which is better?

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Given two clustering $\{C_1, \ldots, C_K\}$ (or c) and $\{C'_1, \ldots, C'_K\}$ (or c')

How do we decide which is better?

- points in same cluster are not dissimilar
- points in different clusters are dissimilar

Clustering Criterion

- Minimize total within-cluster dissimilarity

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- Minimize within-cluster variance: $\mathbf{r}_j = \frac{1}{n_j} \sum_{\mathbf{x} \in C_j} \mathbf{x}$

$$M_5 = \sum_{j=1}^{K} \sum_{t \in C_j} \|\mathbf{x}_t - \mathbf{r}_j\|_2^2$$

How different are these criteria?

Clustering Criterion

- minimizing $M_1 \equiv$ maximizing M_2

CLUSTERING

- Multiple clustering criteria all equally valid
- Different criteria lead to different algorithms/solutions
- Which notion of distances or costs we use matter

Lets Build an Algorithm

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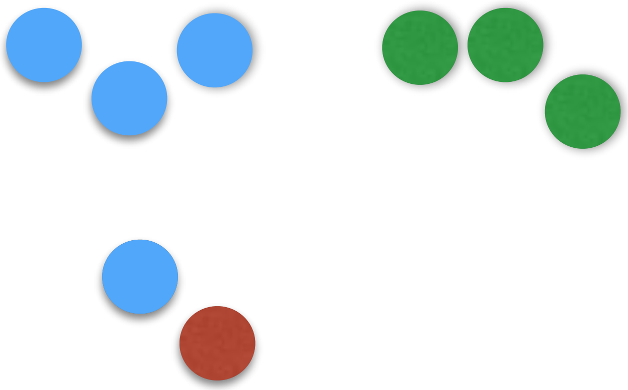

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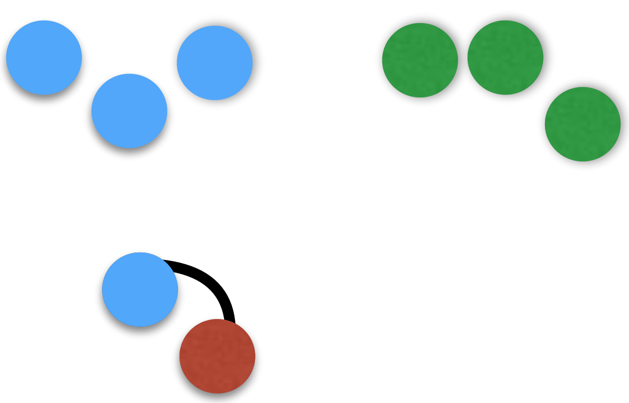

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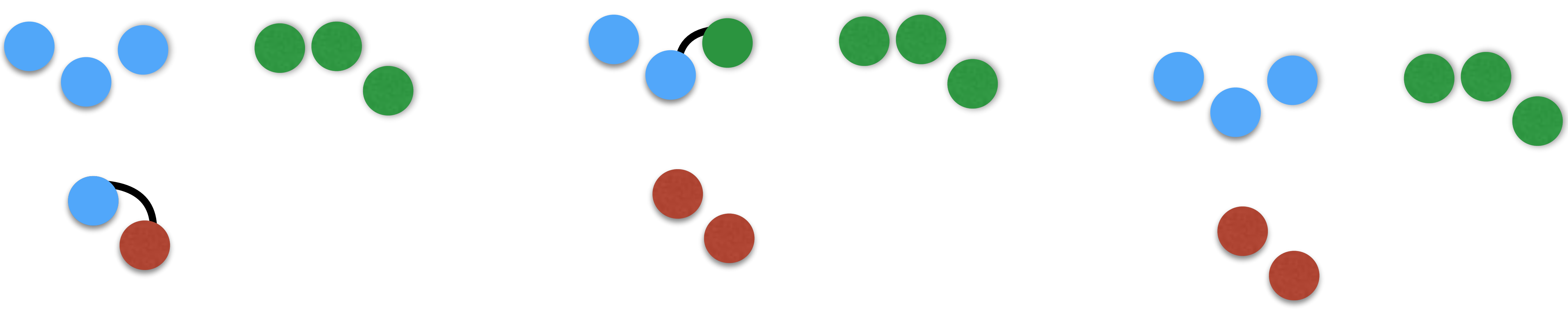

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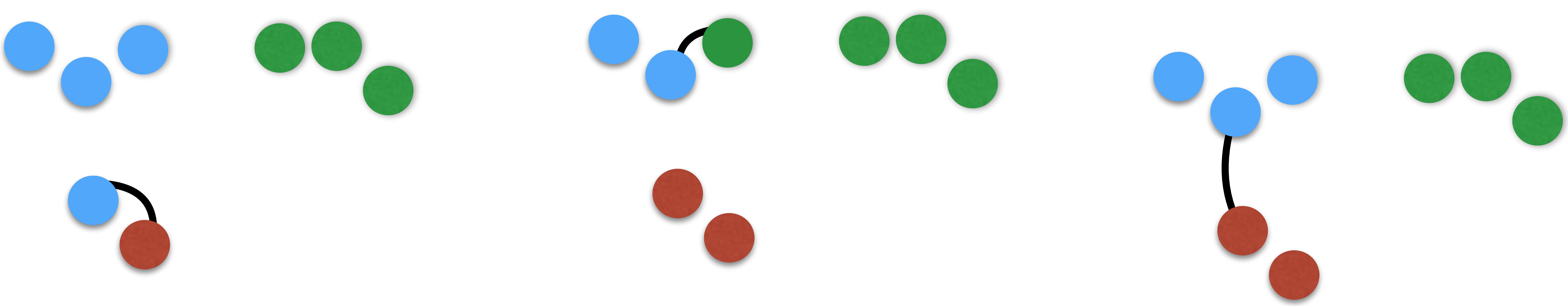

Single Link Clustering

- Initialize n clusters with each point $\mathbf{x}_t$ to its own cluster
- Until there are only K clusters, do
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$$\text{dissimilarity}(C_i, C_j) = \min_{t \in C_i, s \in C_j} \text{dissimilarity}(\mathrm{x}_t, \mathrm{x}_s)$$

Demo

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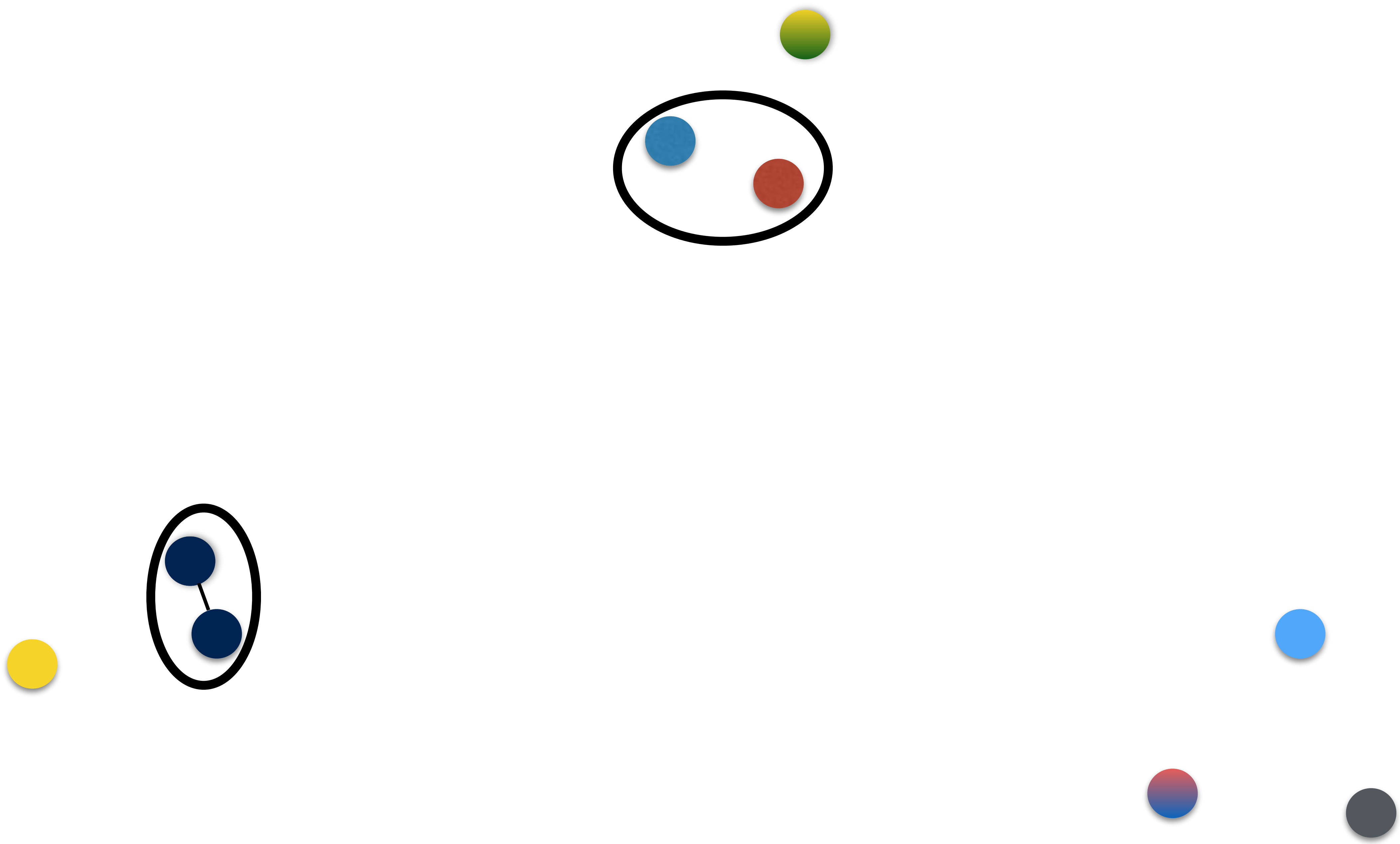

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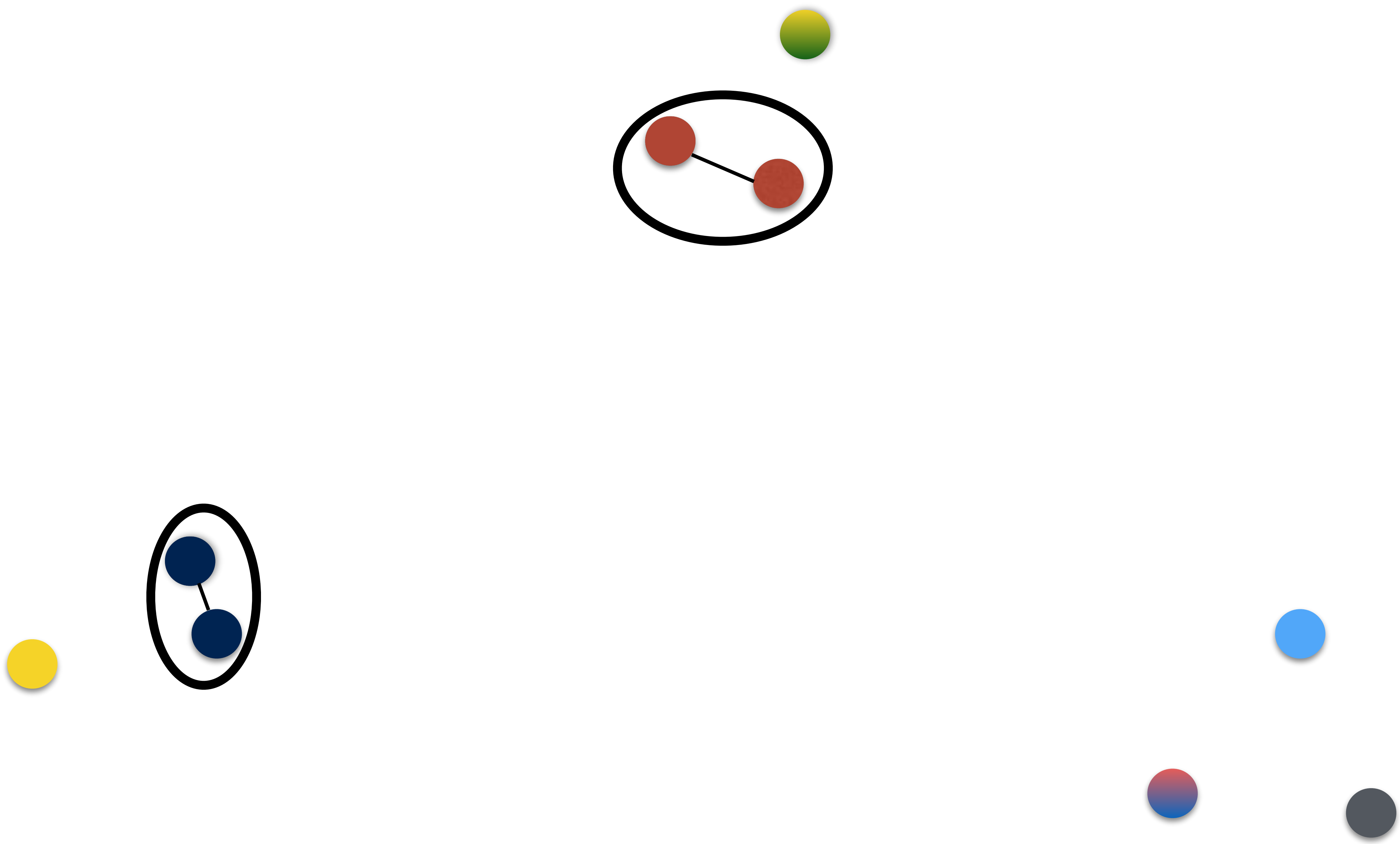

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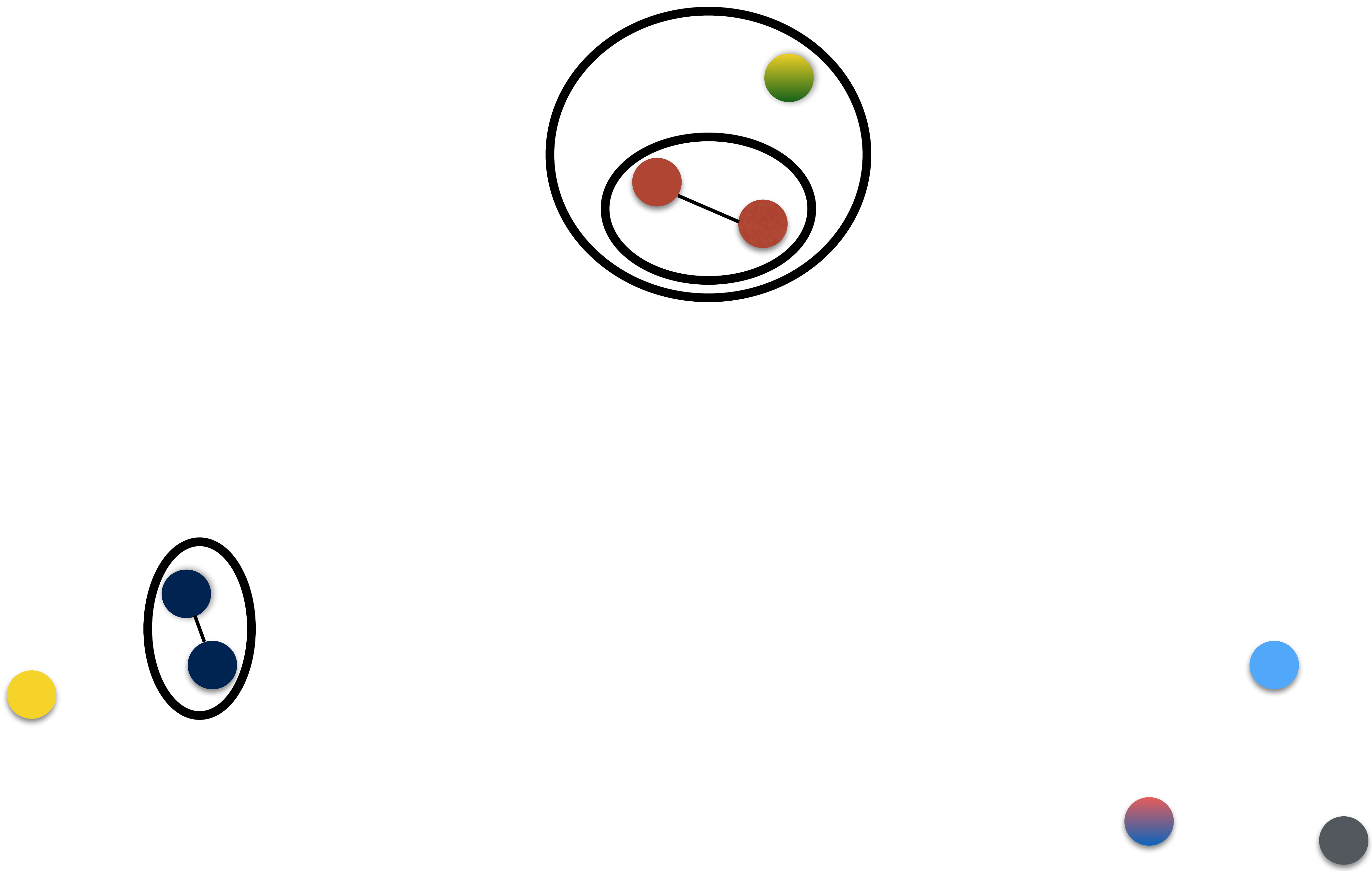

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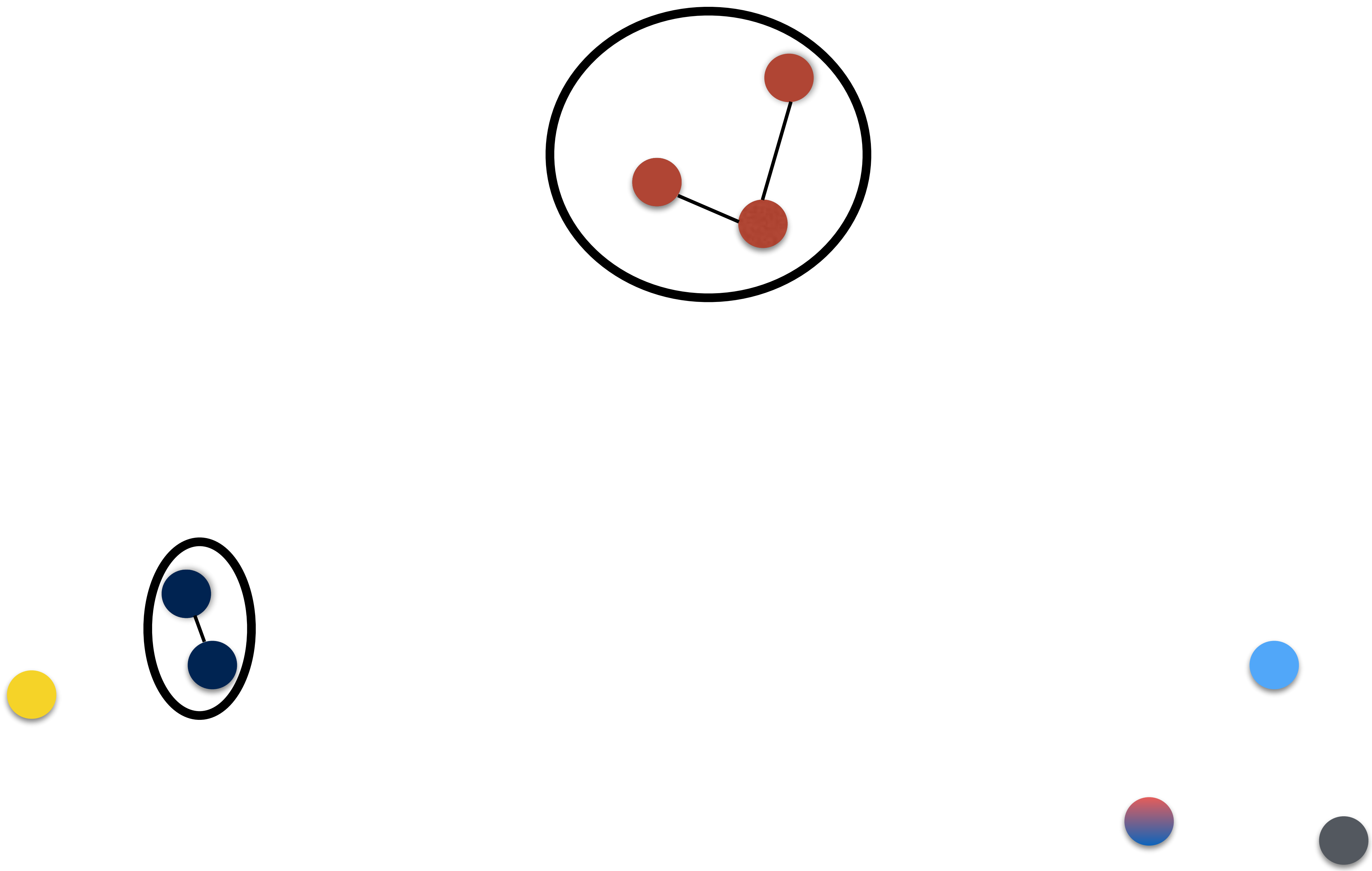

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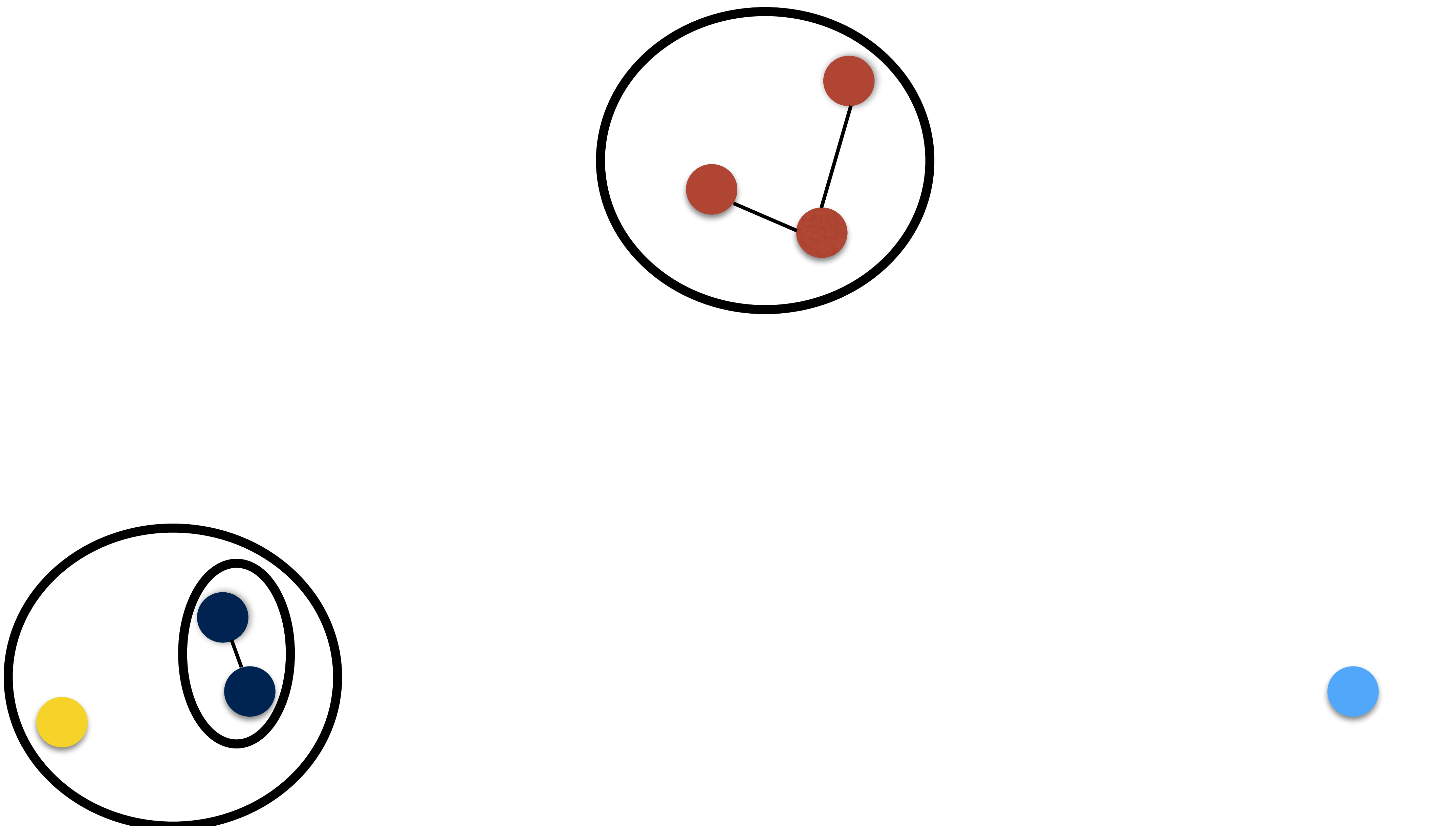

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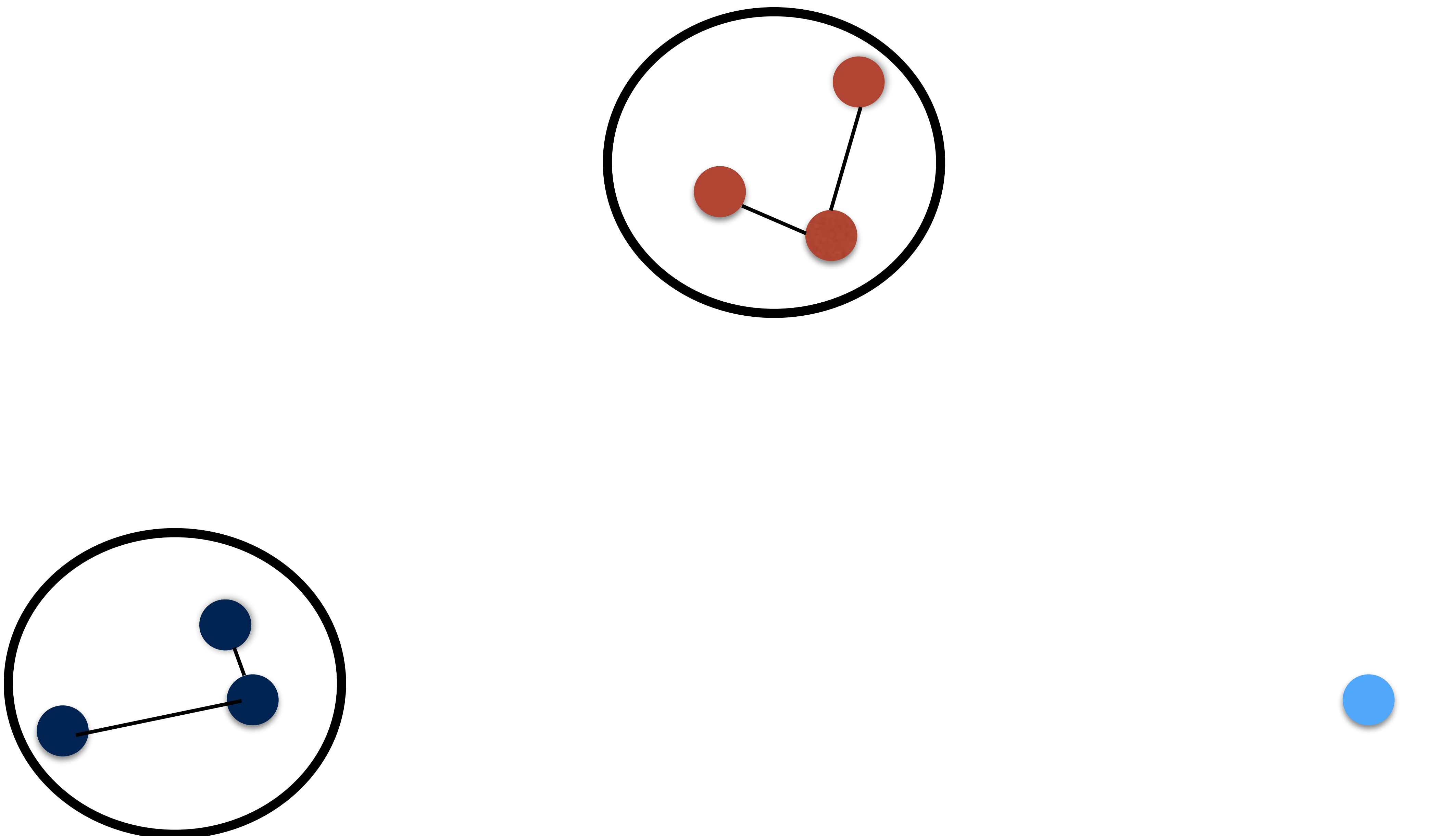

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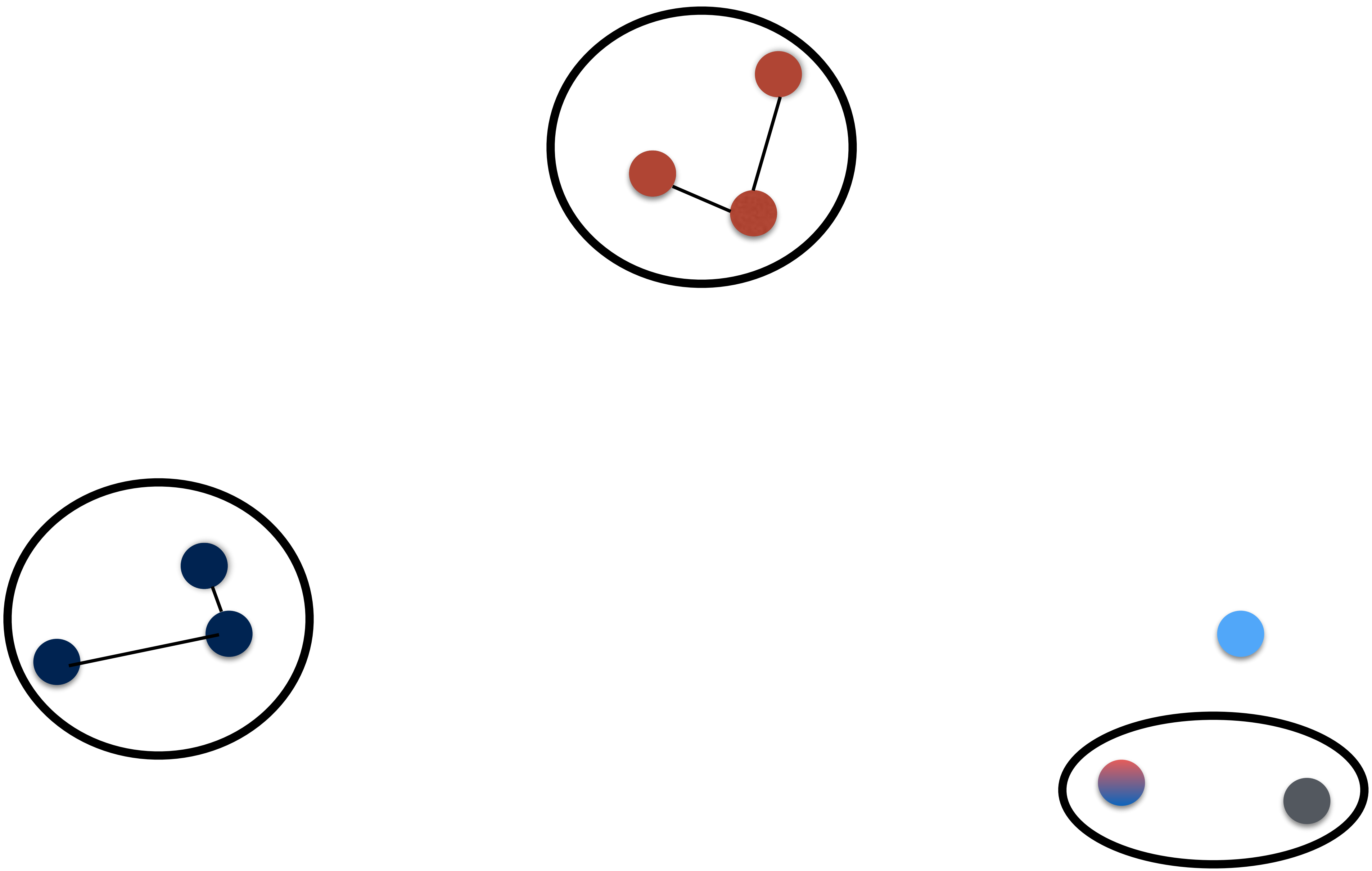

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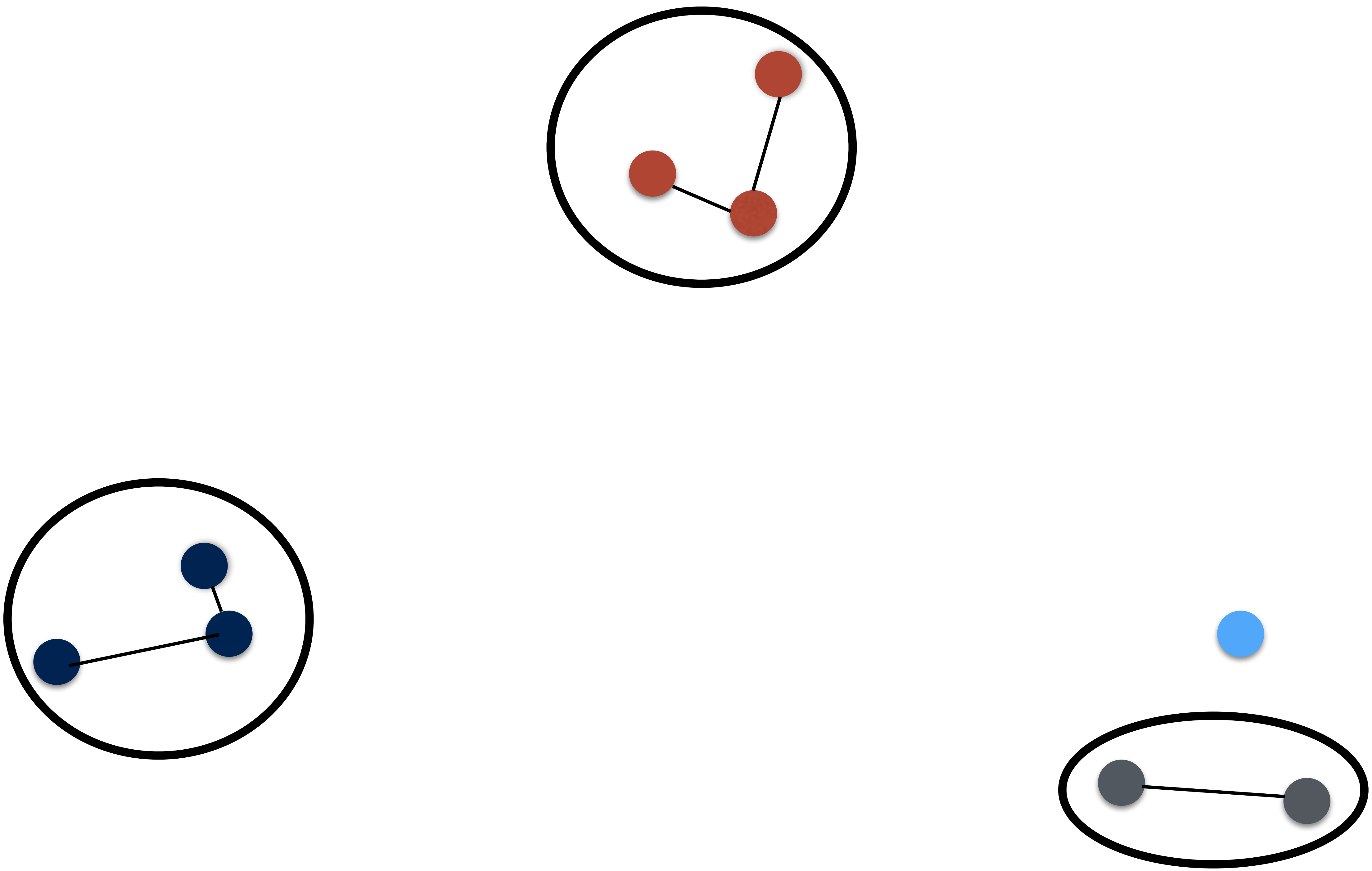

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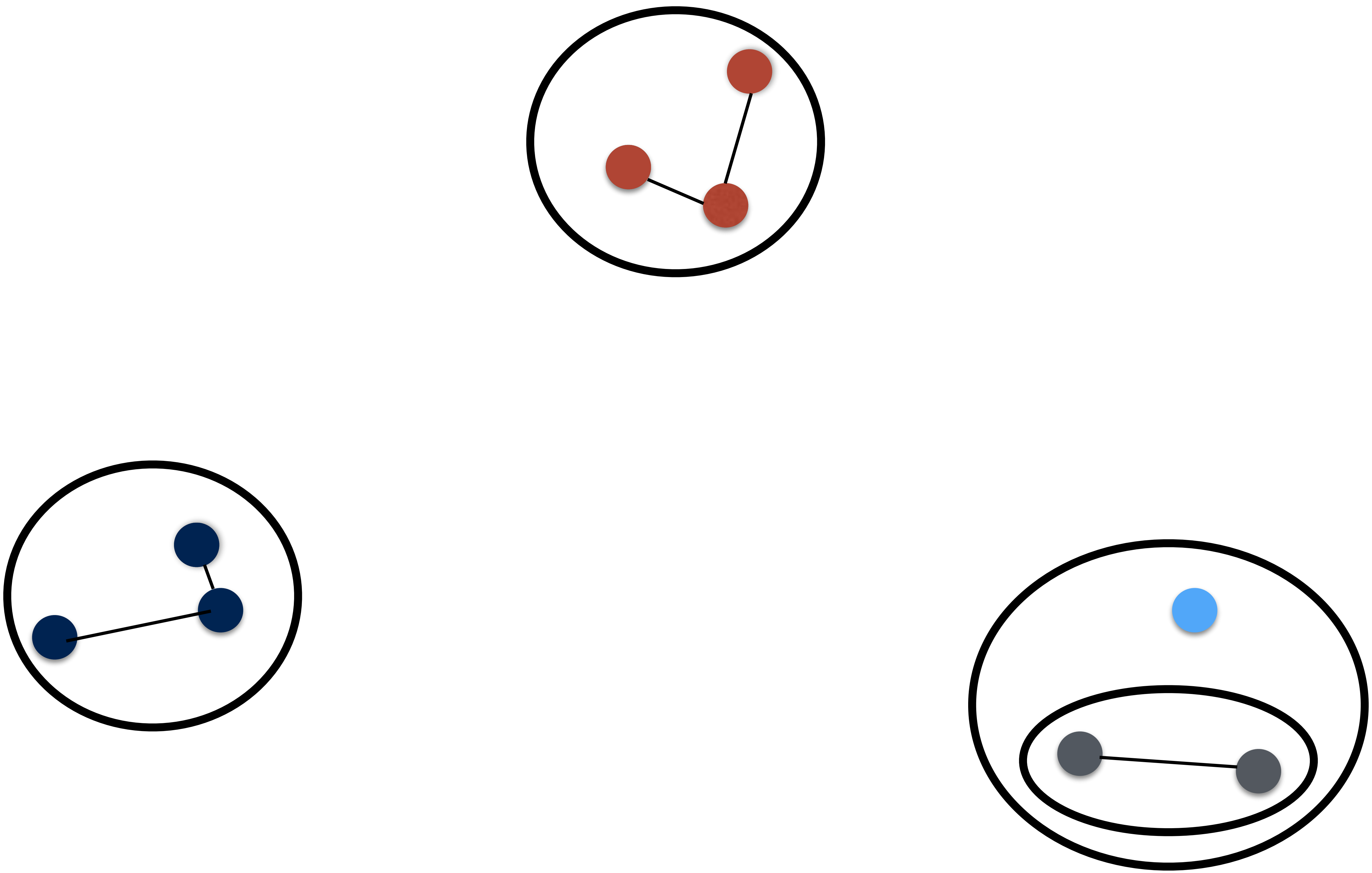

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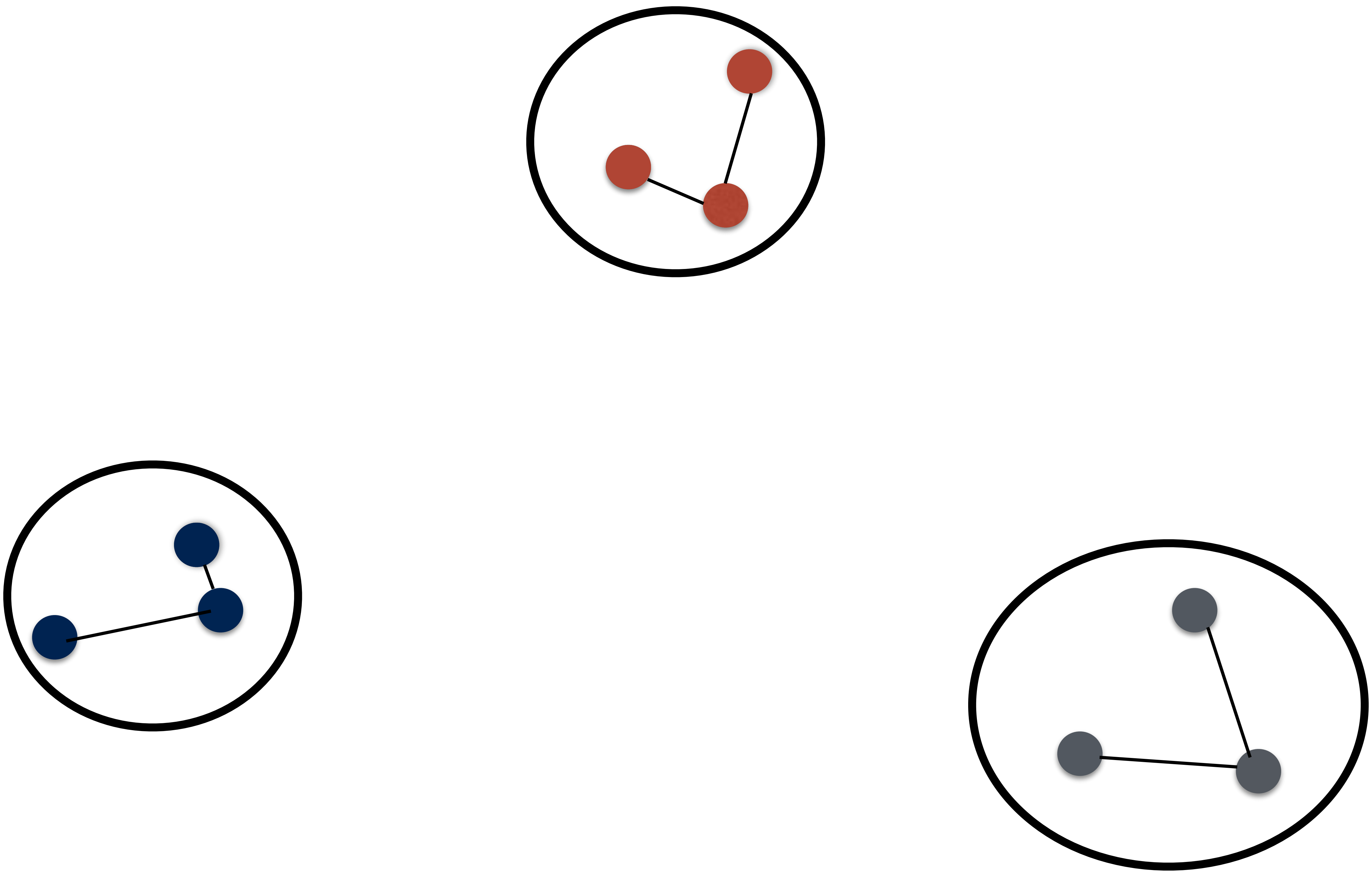

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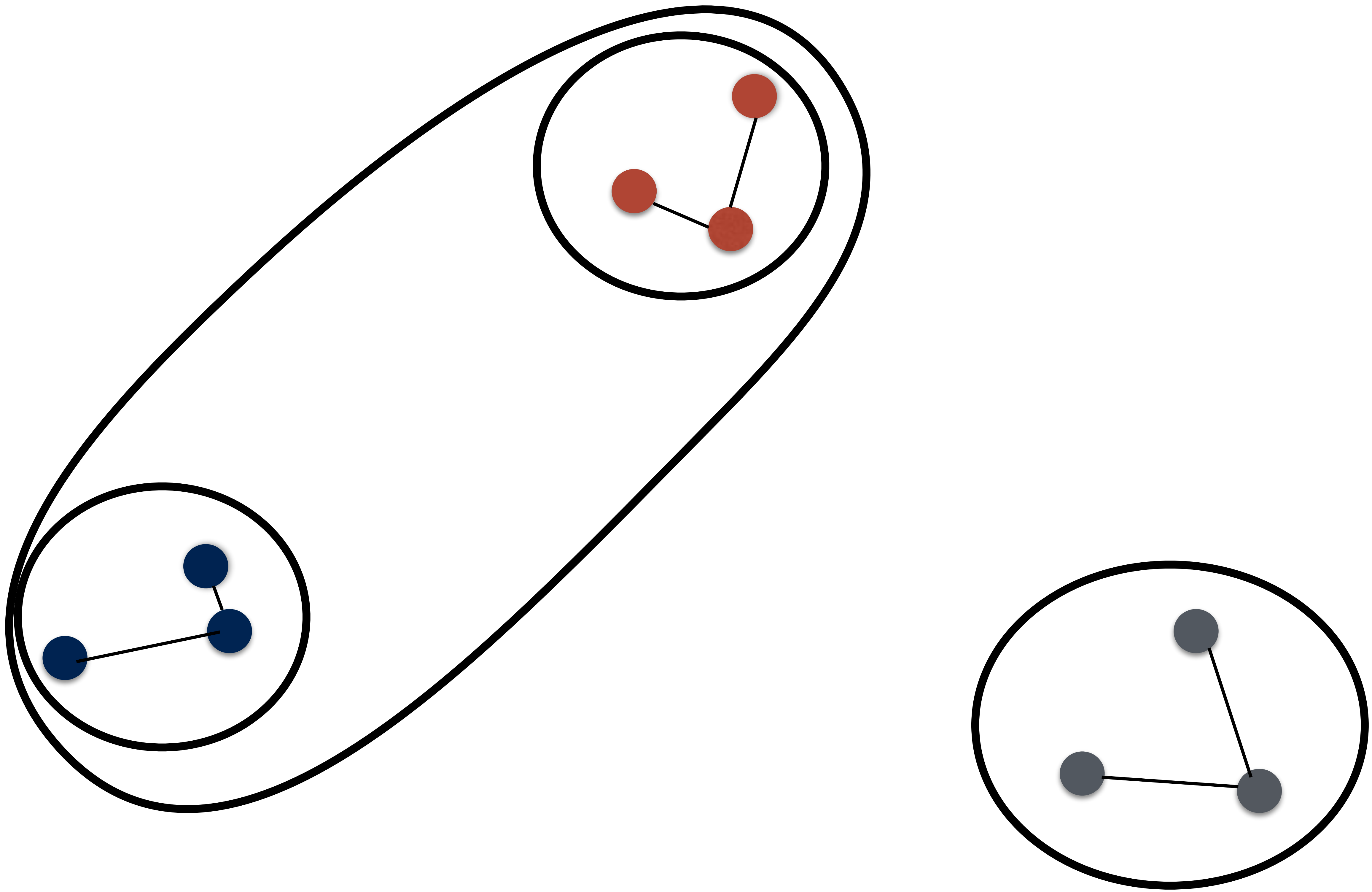

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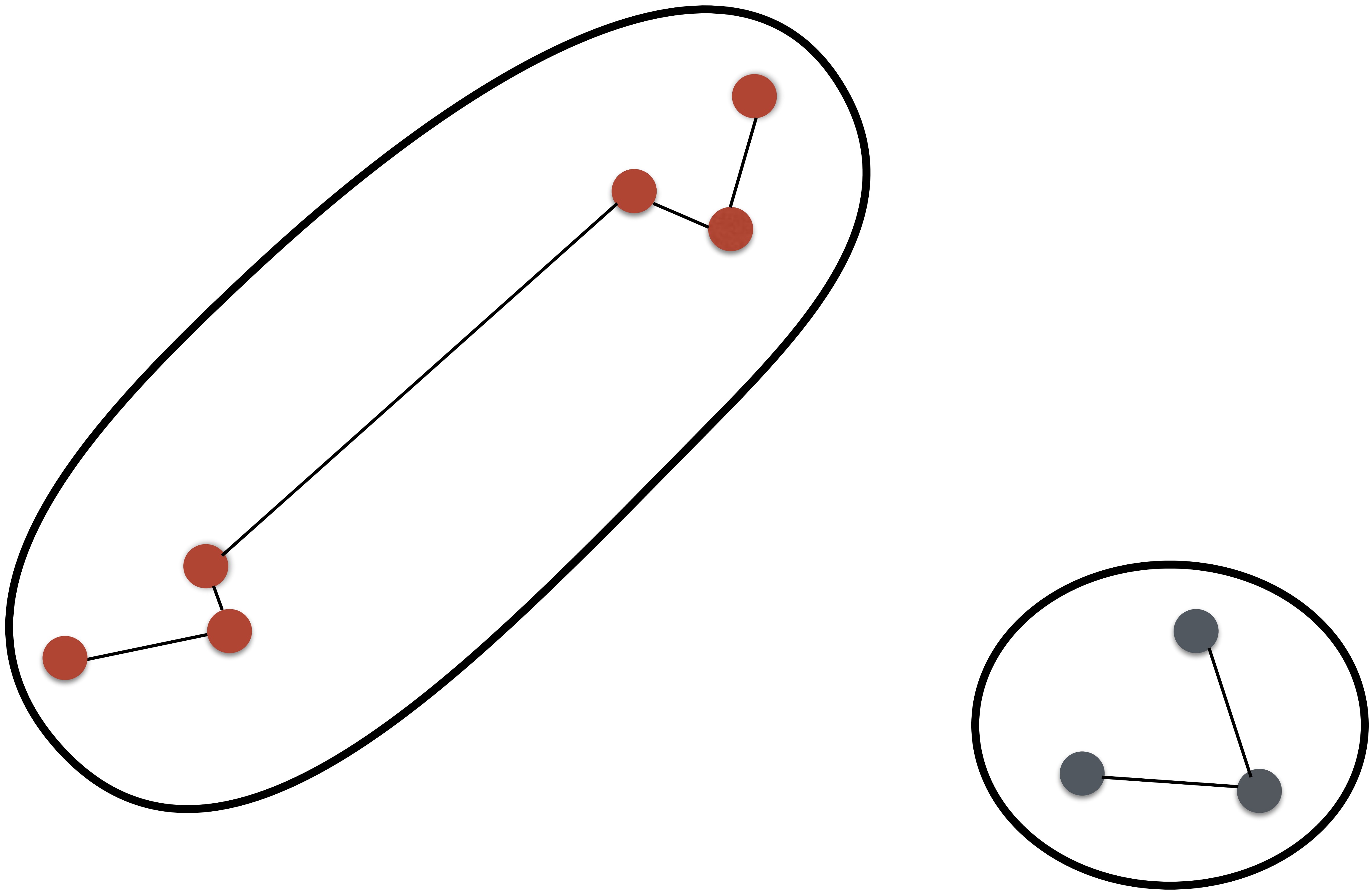

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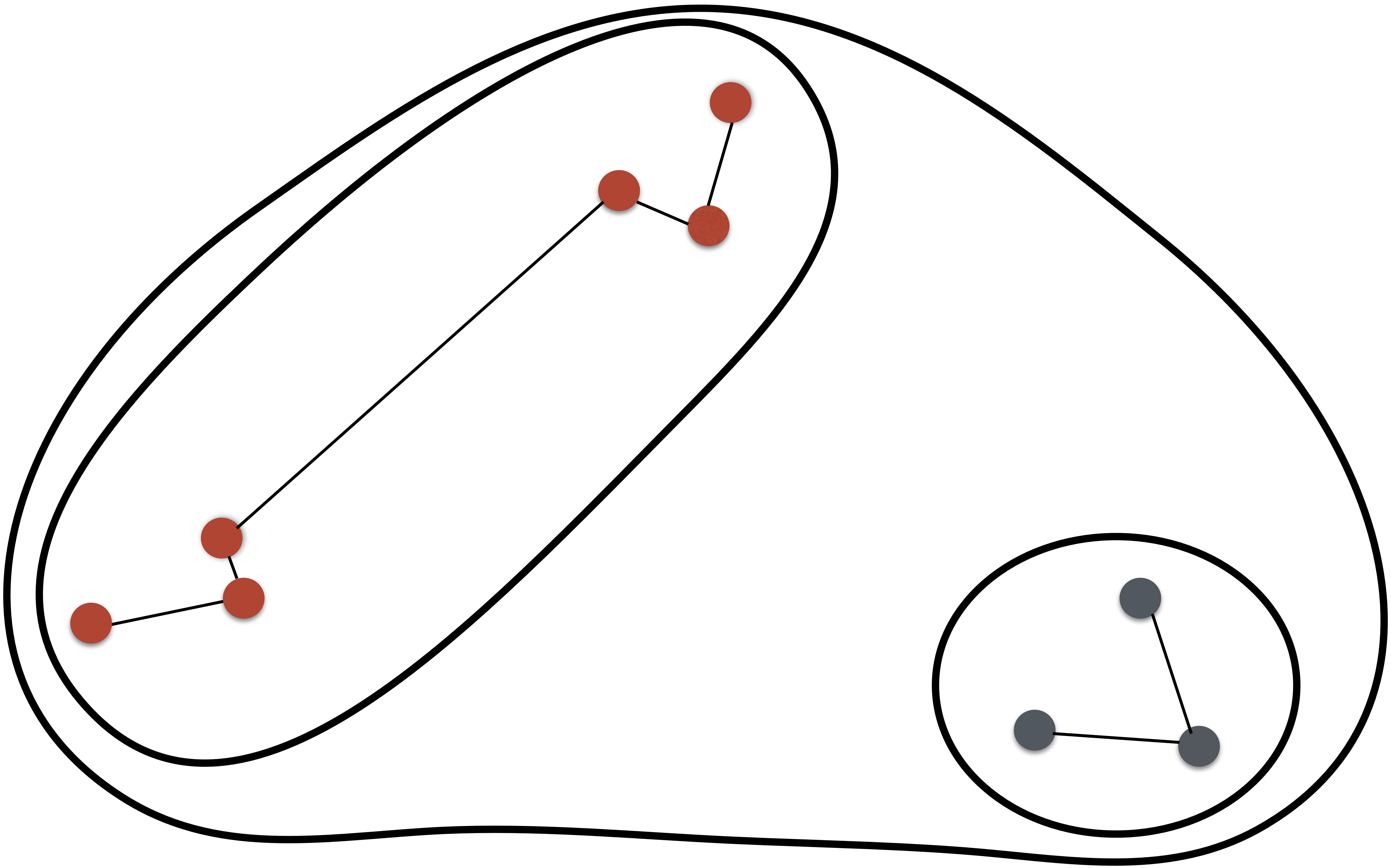

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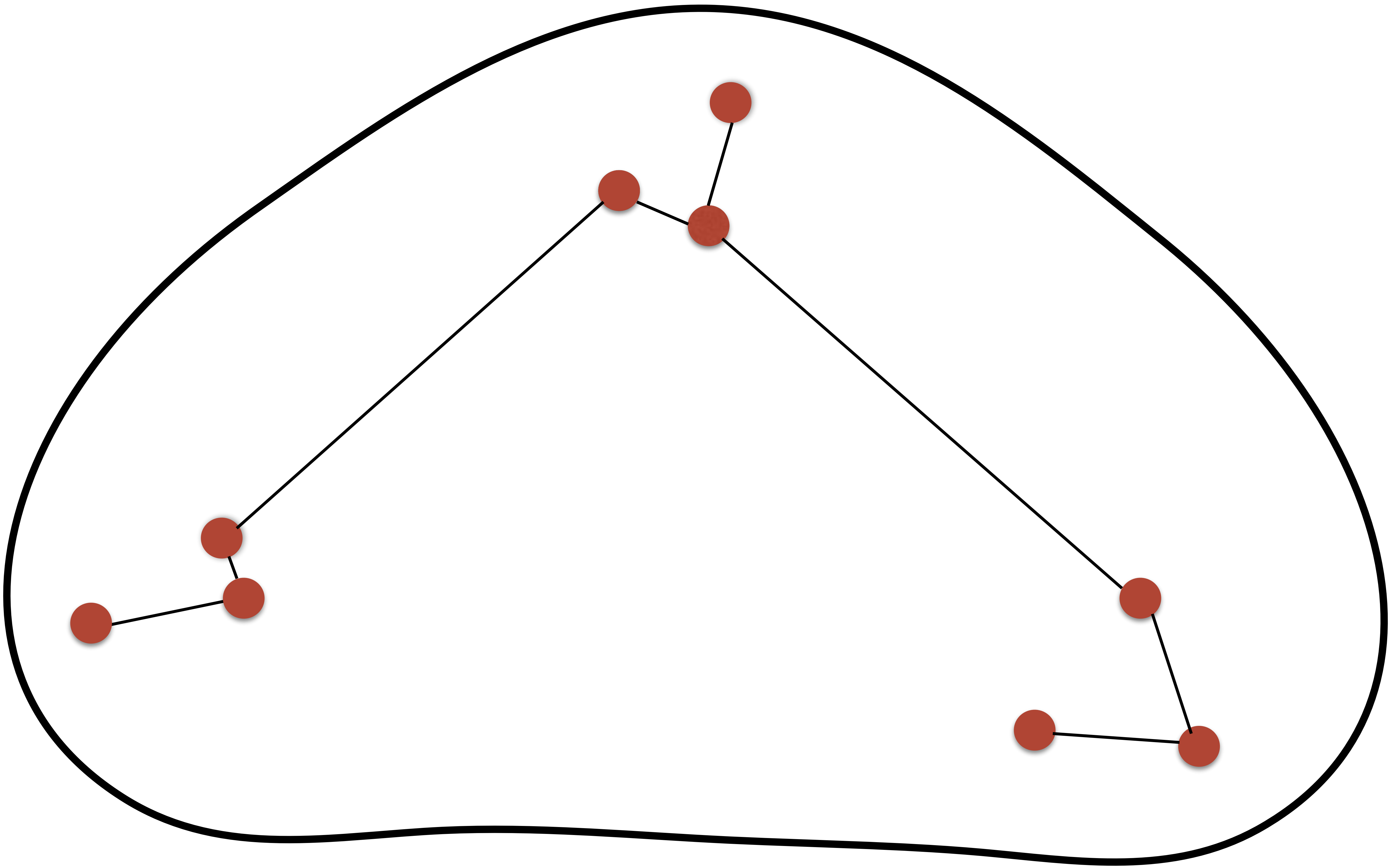

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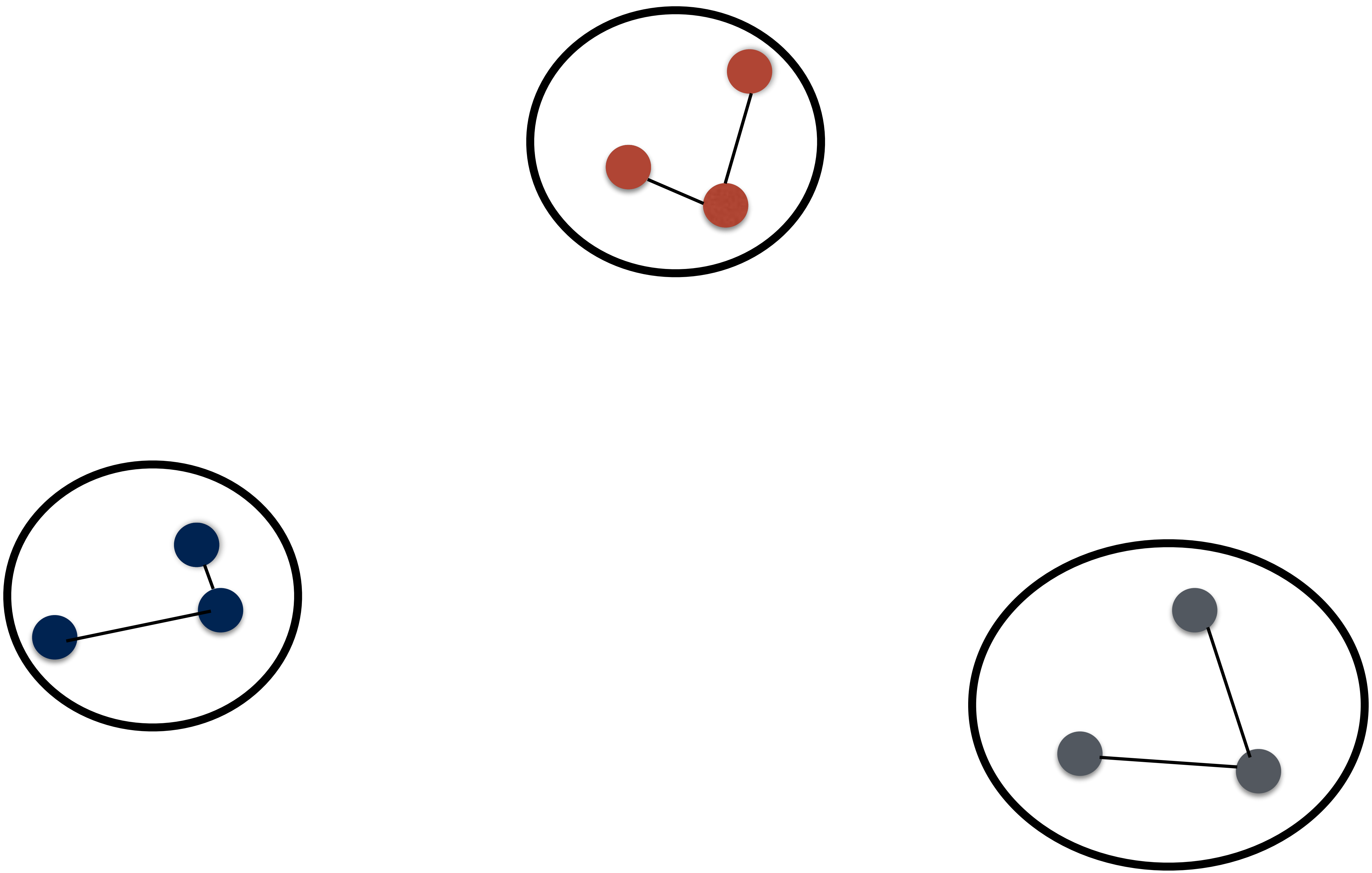

Single Link Objective

Objective for single-link:

$$M_3 = \min_{\mathbf{x}_s, \mathbf{x}_t : c(\mathbf{x}_s) \neq c(\mathbf{x}_t)} \text{dissimilarity}(x_t, x_s)$$

Single link clustering is optimal for above objective!

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Say c is solution produced by single-link clustering

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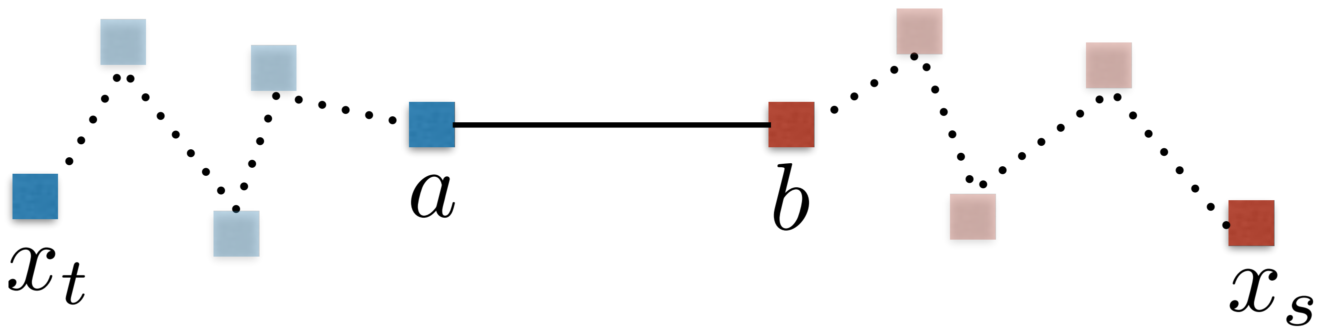

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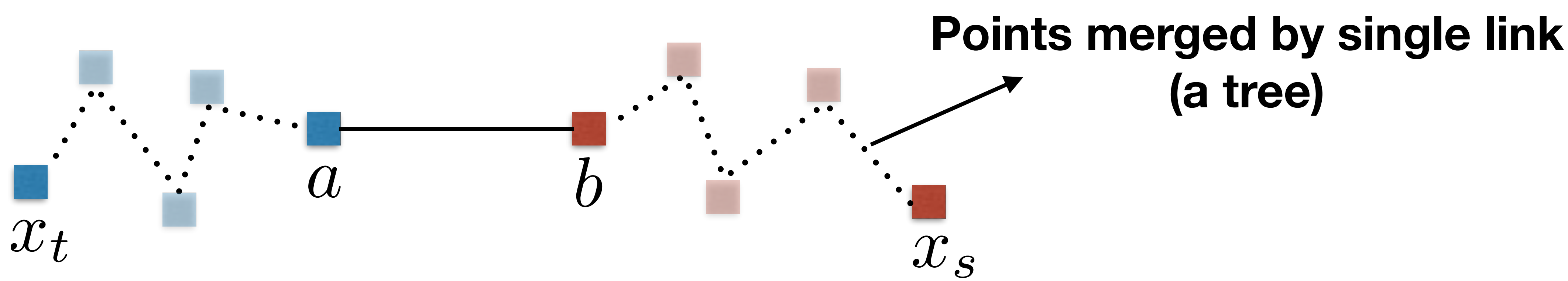

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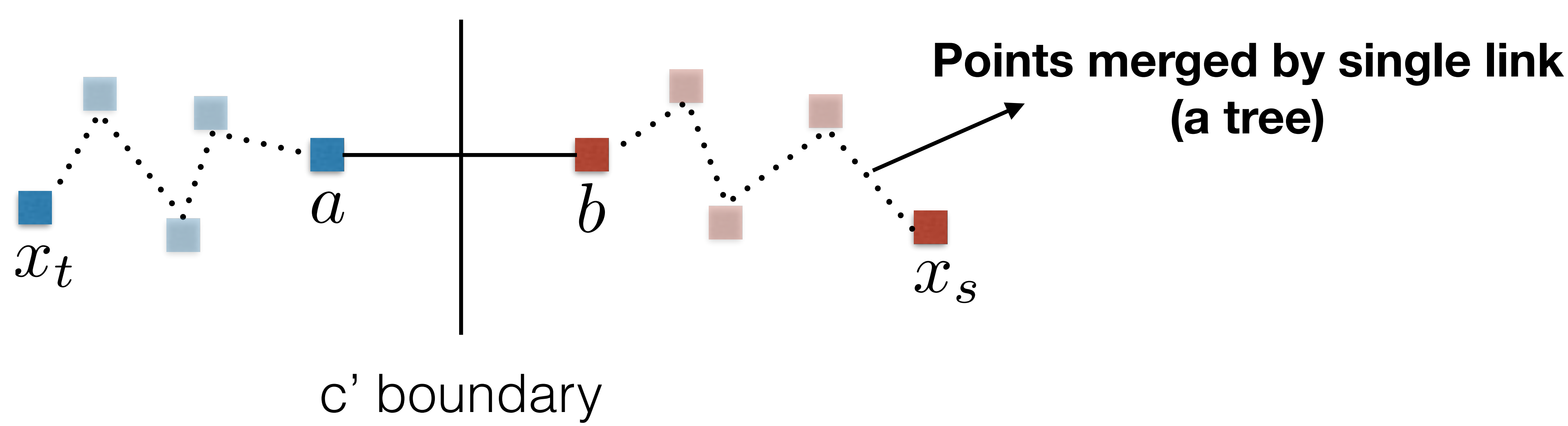

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- Single link is the only one provable optimal
- Linking based on average distance works best in practice

Demo

Clustering Criterion

- Minimize average dissimilarity within cluster

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$$M_5 = \sum_{j=1}^{K} \sum_{t \in C_j} \|\mathbf{x}_t - \mathbf{r}_j\|_2^2$$

Clustering Criterion

- minimizing $M_5 \equiv$ minimizing M_6

What is the Algorithm for this?