1 ISOMAP (Multidimensional Scaling)

Question 1: Say the $n \times n$ matrix D consists of squared (Euclidean) distance between all pairs of the n points. That is, $D_{i,j}$ is the squared distance between the i'th and j'th data point. That is, if the i'th data point is given by vector $\mathbf{y}_i$ then

$$D_{i,j} = \|\mathbf{y}_i - \mathbf{y}_j\|_2^2$$

Hint: assume w.l.o.g. that the points are centered, that is there sum is the 0 vector. Notice that this is an assumption we can make for free because even if we center y's by subtracting the mean, the inter point distances still remains the same. In this case show that:

$$\mathbf{y}_i^\top \mathbf{y}_j = \frac{1}{2}\left(\frac{1}{n}\sum_{j=1}^{n} D_{i,j} + \frac{1}{n}\sum_{i=1}^{n} D_{i,j} - \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} D_{i,j} - D_{i,j}\right)$$

2 t-SNE

Question 2: Given points $\mathbf{x}_1, \ldots, \mathbf{x}_n$, for any $t, s \in [n]$, let

$$p_{t \to s} = \frac{\exp(-\frac{\|\mathbf{x}_s - \mathbf{x}_t\|^2}{2\sigma^2})}{\sum_{u \neq t} \exp(-\frac{\|\mathbf{x}_u - \mathbf{x}_t\|^2}{2\sigma^2})}$$

Now define $P_{s,t} = \frac{p_{t \to s} + p_{s \to t}}{2n}$ and assume $P_{t,t} = 0$ for any t. Show that P is a valid probability distribution over $[n] \times [n]$.