When poll is active respond at **PollEv.com/weichiuma**

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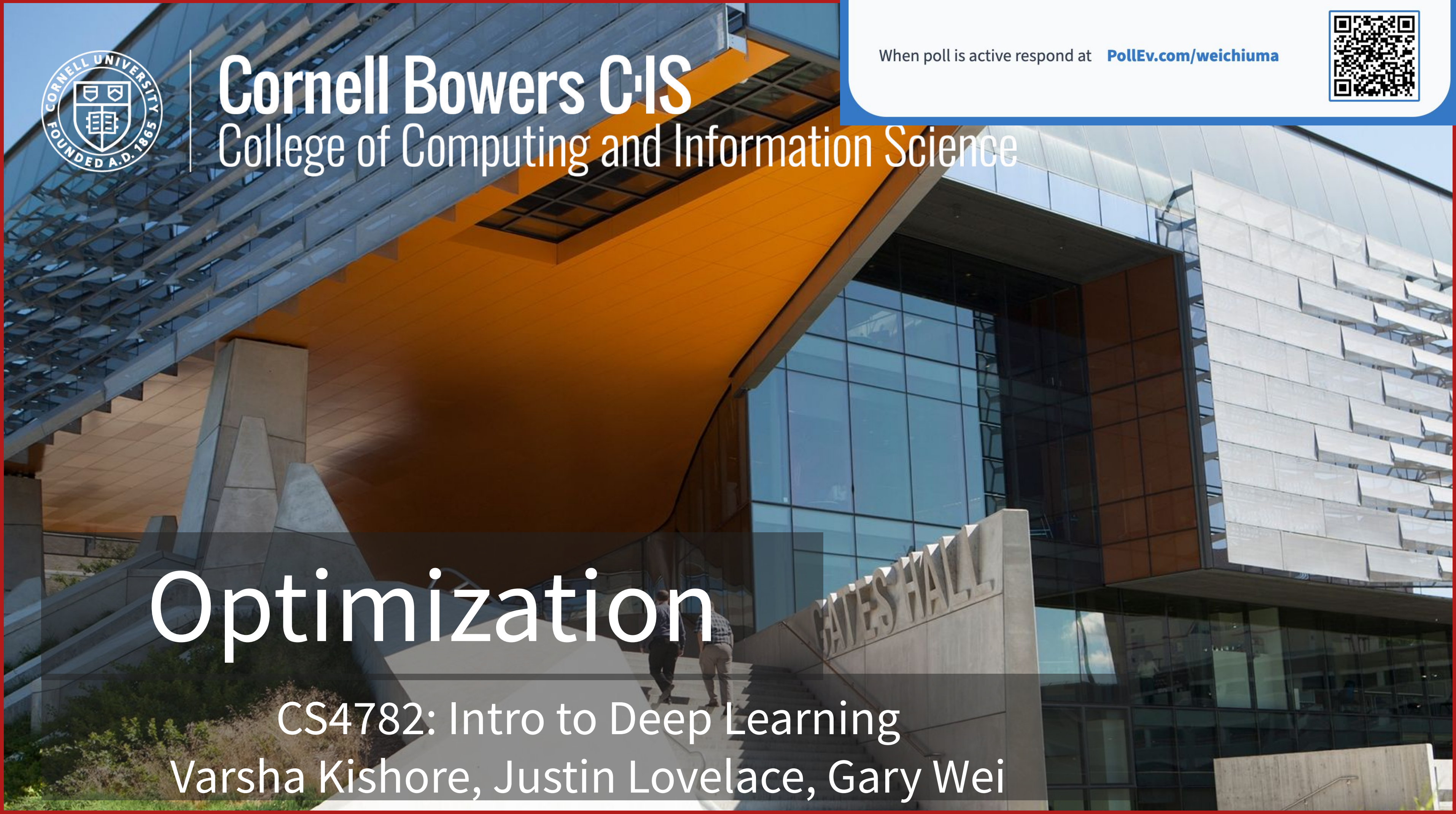
College of Computing and Information Science



Optimization

CS4782: Intro to Deep Learning

Varsha Kishore, Justin Lovelace, Gary Wei

Before we start:

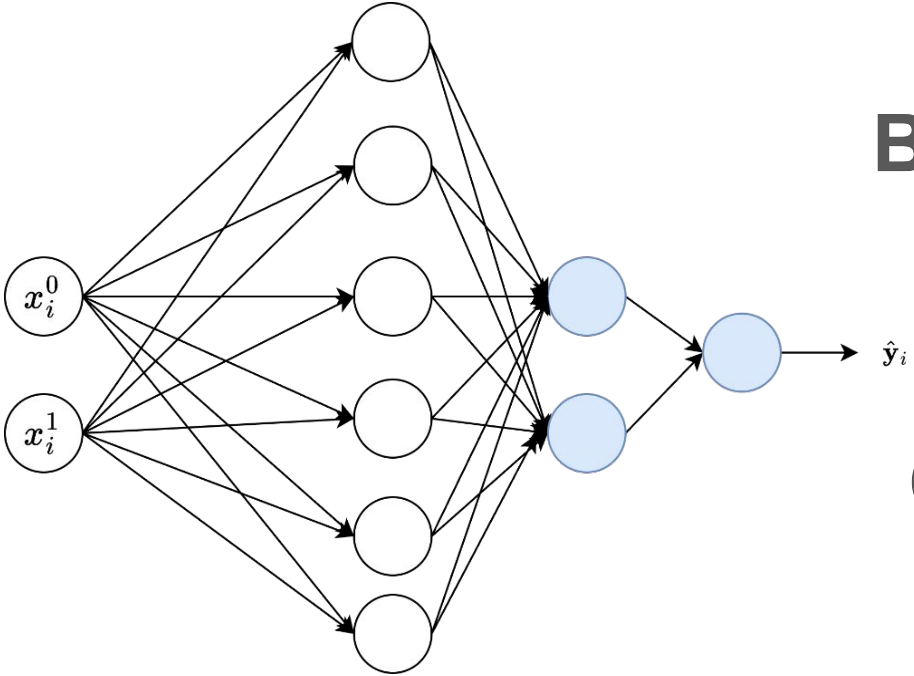
Look at recap quizzes 1,2

Discuss with your neighbors

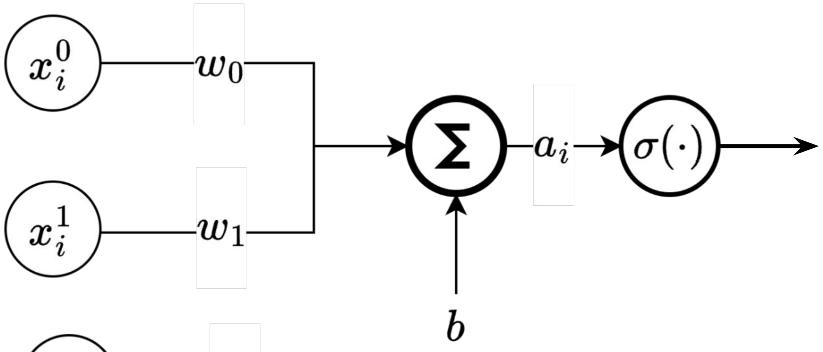
Choose option A, B, C

When poll is active respond at **PollEv.com/weichiuma**

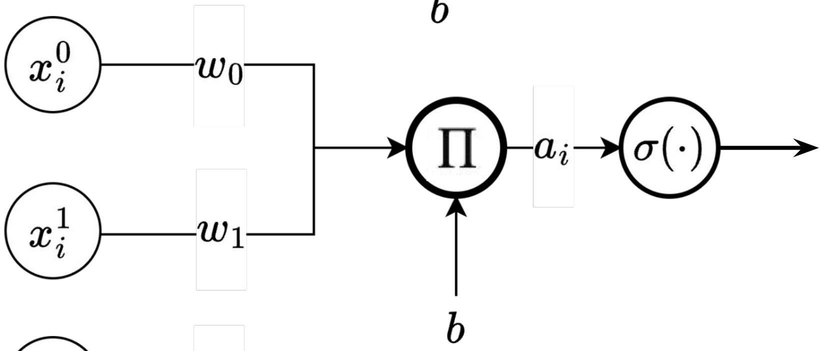
What's inside one of the circles?



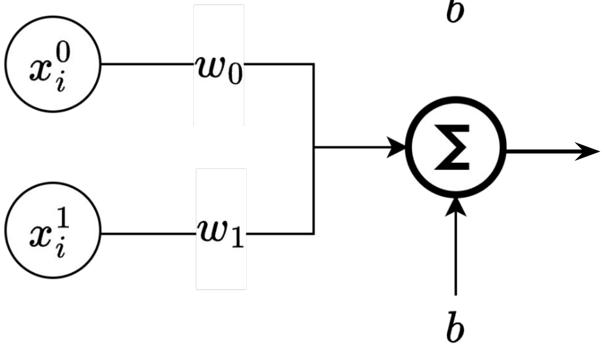
A.



B.

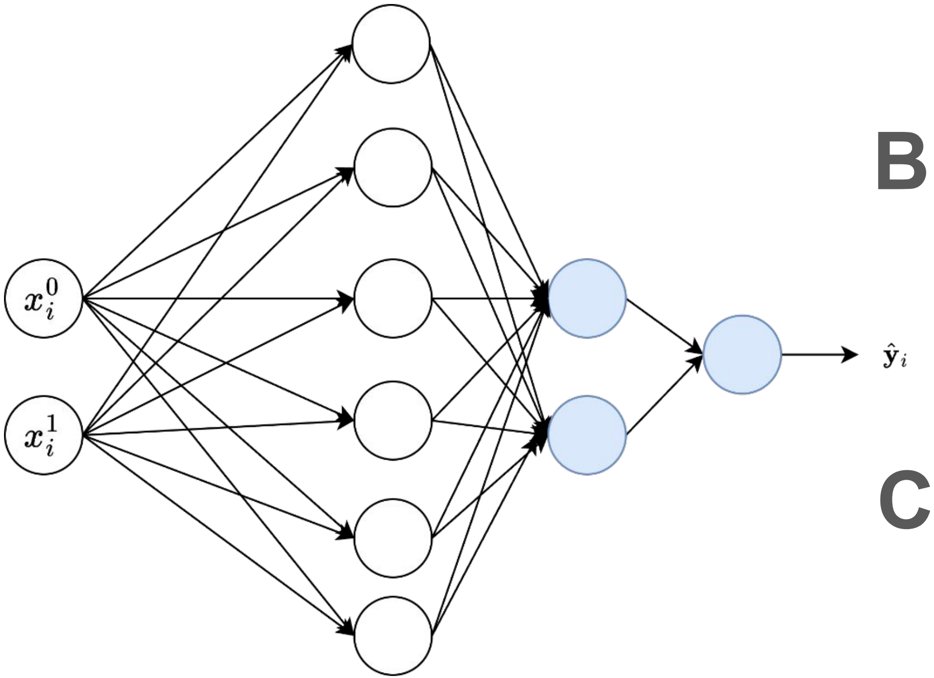


C.

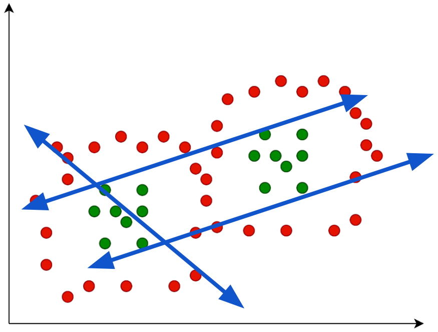


When poll is active respond at PollEv.com/weichiuma

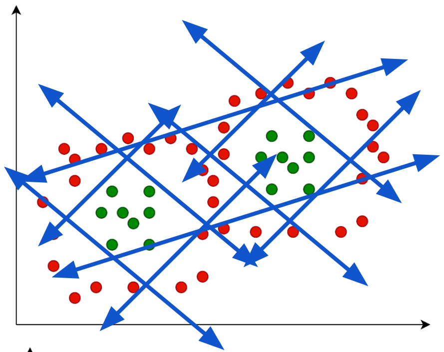
Which decision boundary CANNOT be learned by this network?



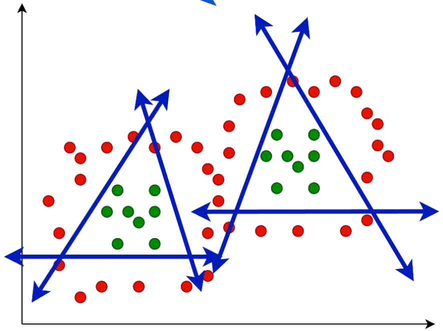
A.



B.



C.



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Course Announcement

- If you are in **5782**
  - Paper quizzes are mandatory (10%)

- If you are in **4782**
  - Paper quizzes are optional
  - If you do them, we will use the better grade with or without quizzes

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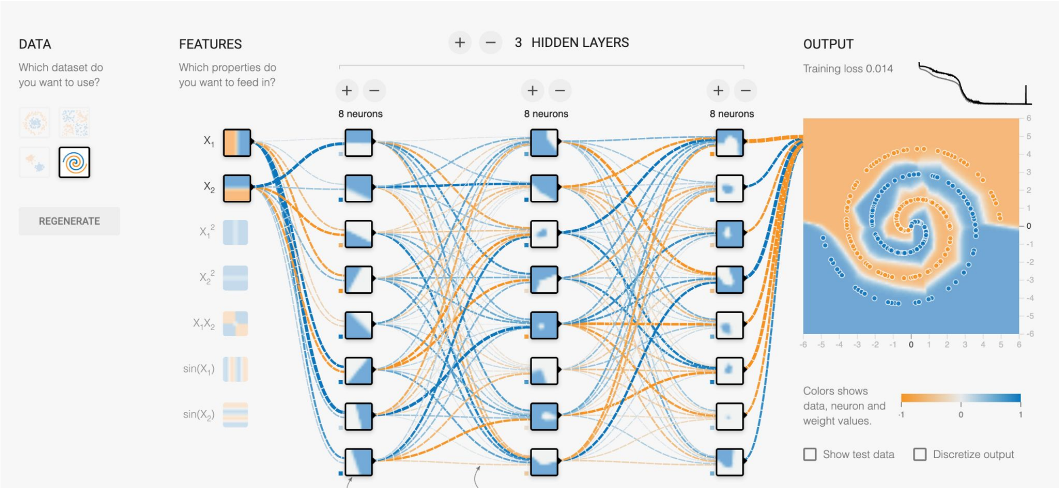
Agenda

- Backpropagation
- Optimizers
    - Gradient Descent
    - Stochastic Gradient Descent
    - SGD w. Momentum
    - AdaGrad
    - RMSProp
    - Adam
- Learning rate scheduling



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Calculus Review: The Chain Rule

Lagrange's Notation: If $h(x) = f(g(x))$, then $h' = f'(g(x))g'(x)$

Leibniz's Notation: If $z = h(y), y = g(x)$, then $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$

Calculus Review: The Chain Rule

Lagrange's Notation: If $h(x) = f(g(x))$, then $h' = f'(g(x))g'(x)$

Leibniz's Notation: If $z = h(y), y = g(x)$, then $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$

Example: If $z = \ln(y), y = x^2$, then

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$$

$$= (\frac{1}{y})(2x) = (\frac{1}{x^2})(2x) = \frac{2}{x}$$

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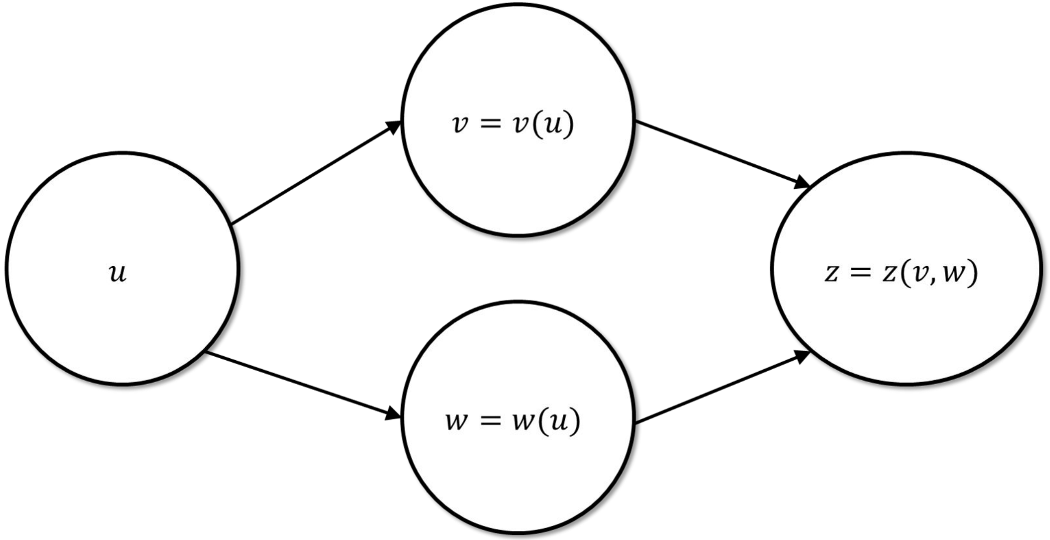
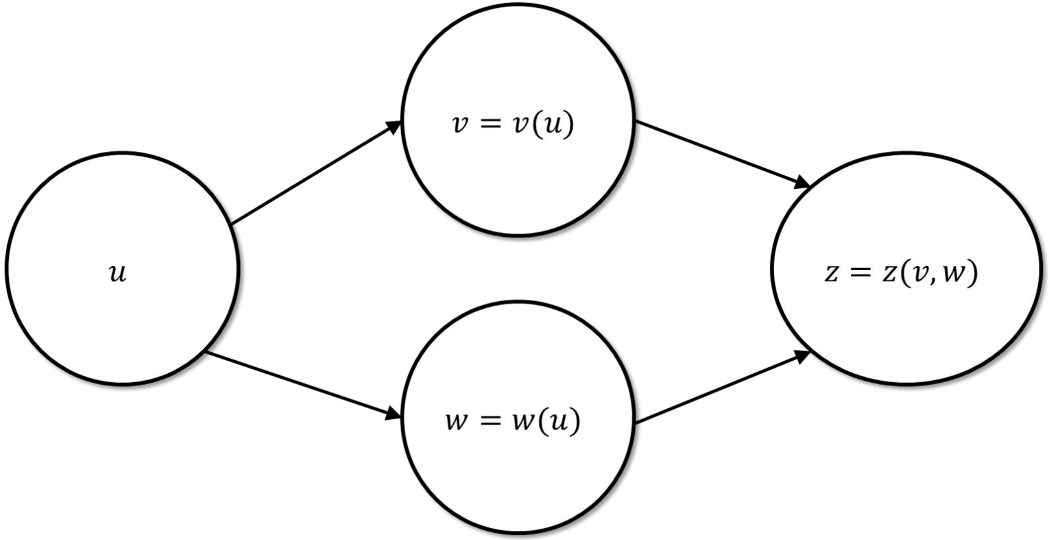
Multivariate Chain Rule



https://windowsontheory.org/2020/11/03/yet-another-backpropagation-tutorial/

Multivariate Chain Rule

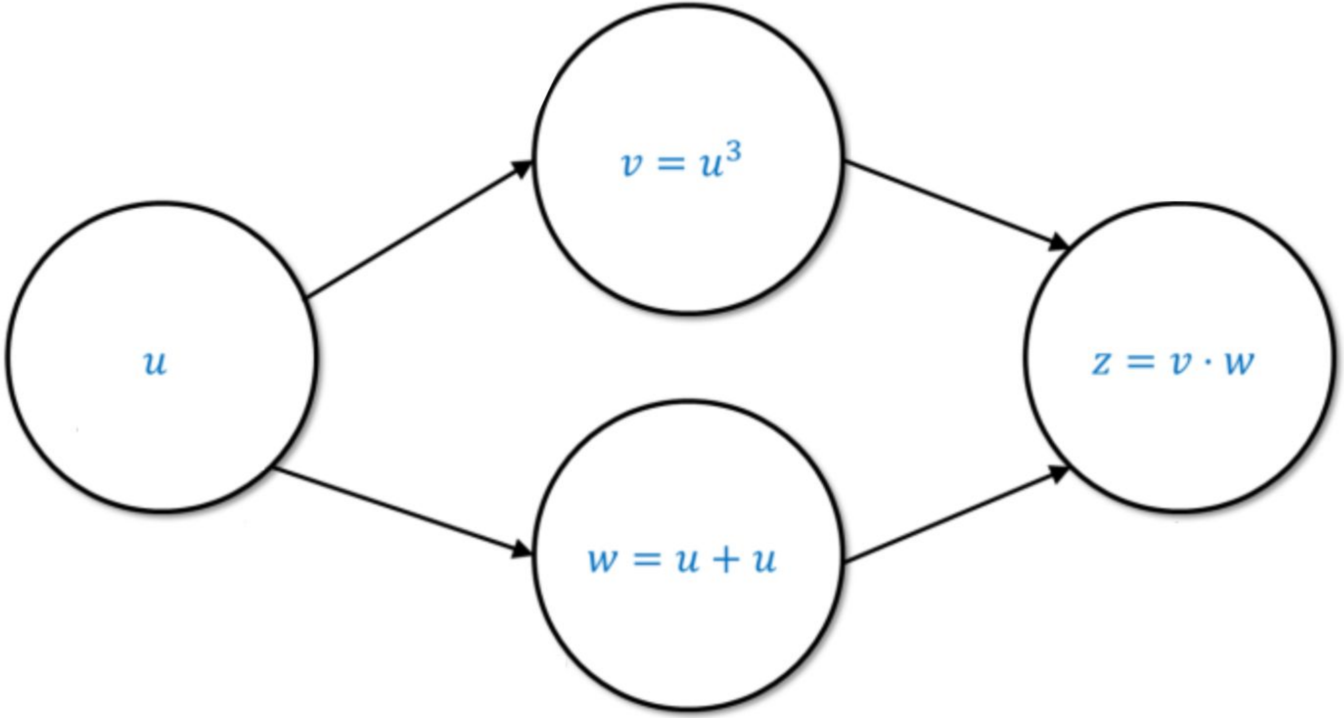


If $f(u)$ is $z = f(v(u), w(u))$, then

$$\frac{\partial f}{\partial u} = (\frac{\partial v}{\partial u}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial u}\frac{\partial z}{\partial w})$$

https://windowsontheory.org/2020/11/03/yet-another-backpropagation-tutorial/

Backpropagation- An Example



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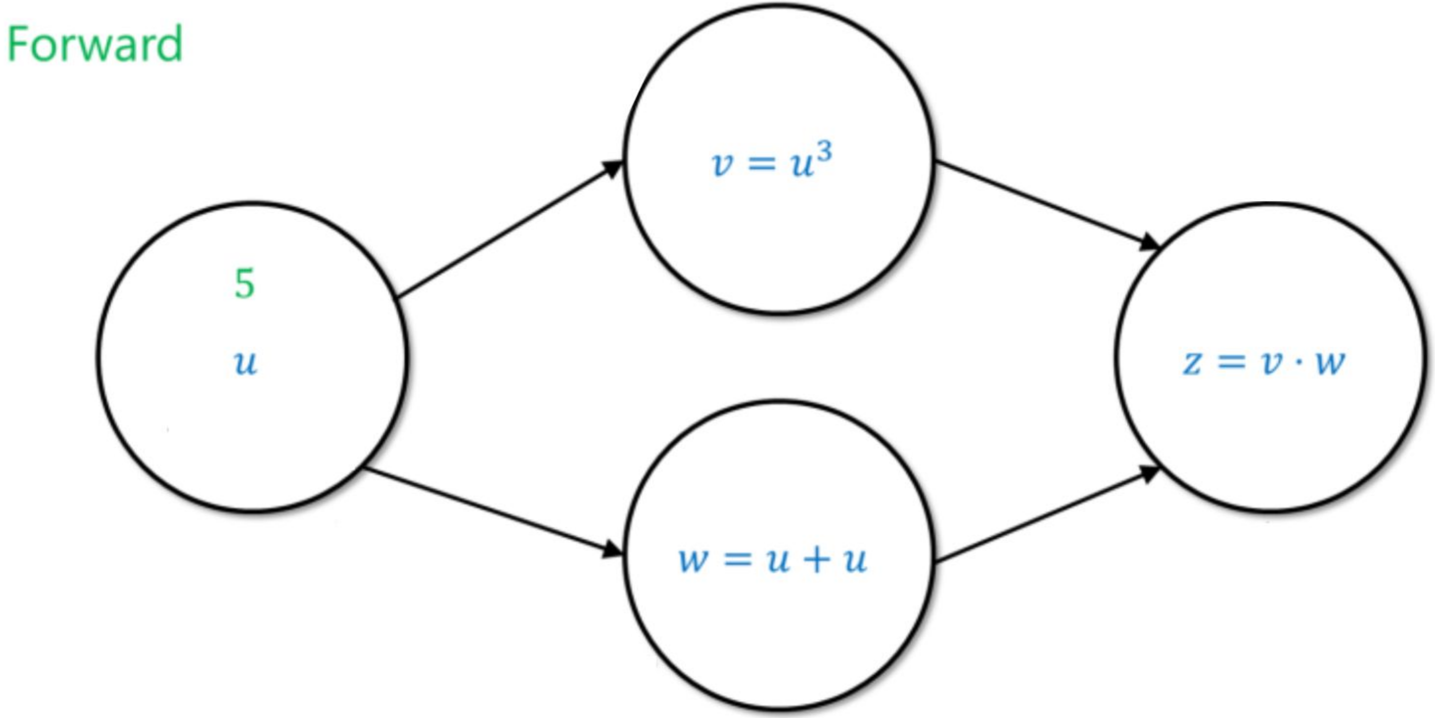
Backpropagation- An Example



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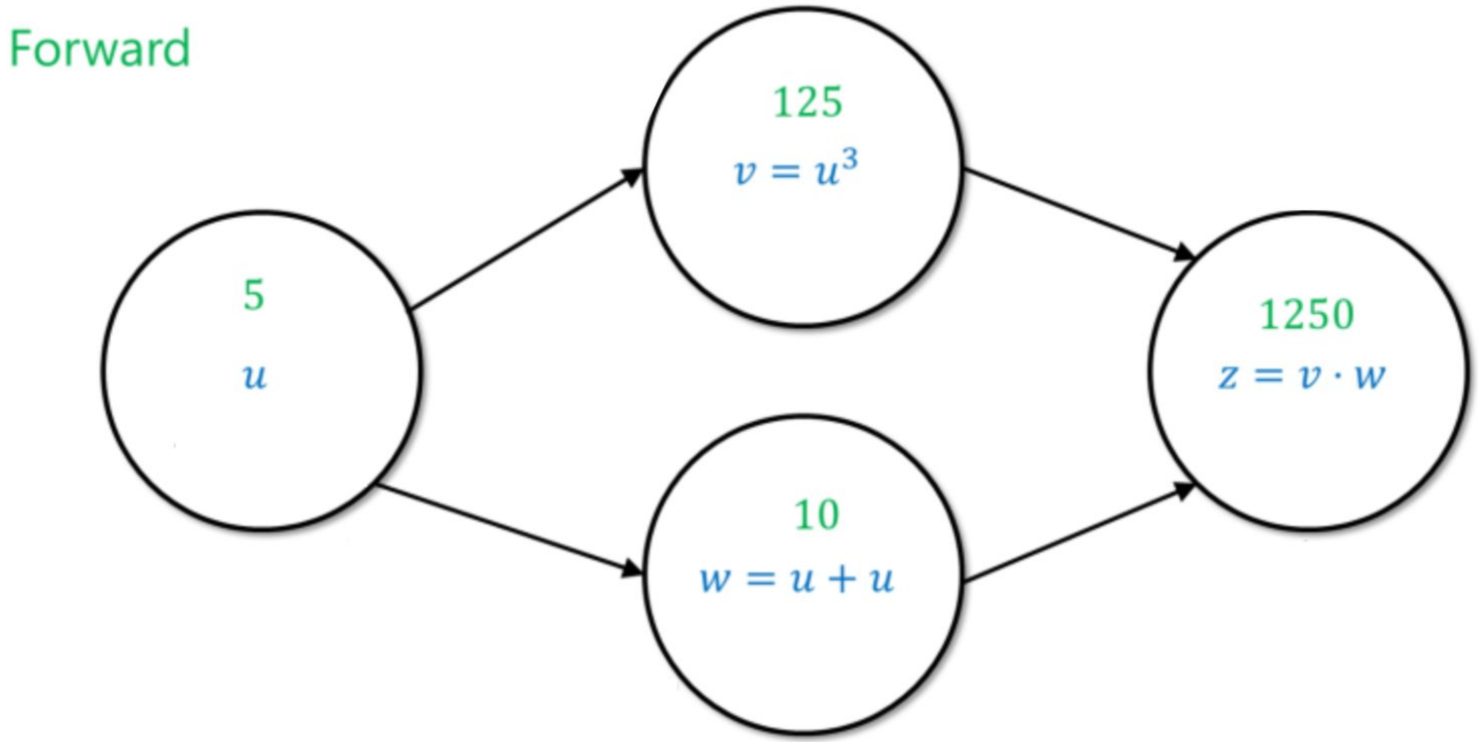
Backpropagation- An Example



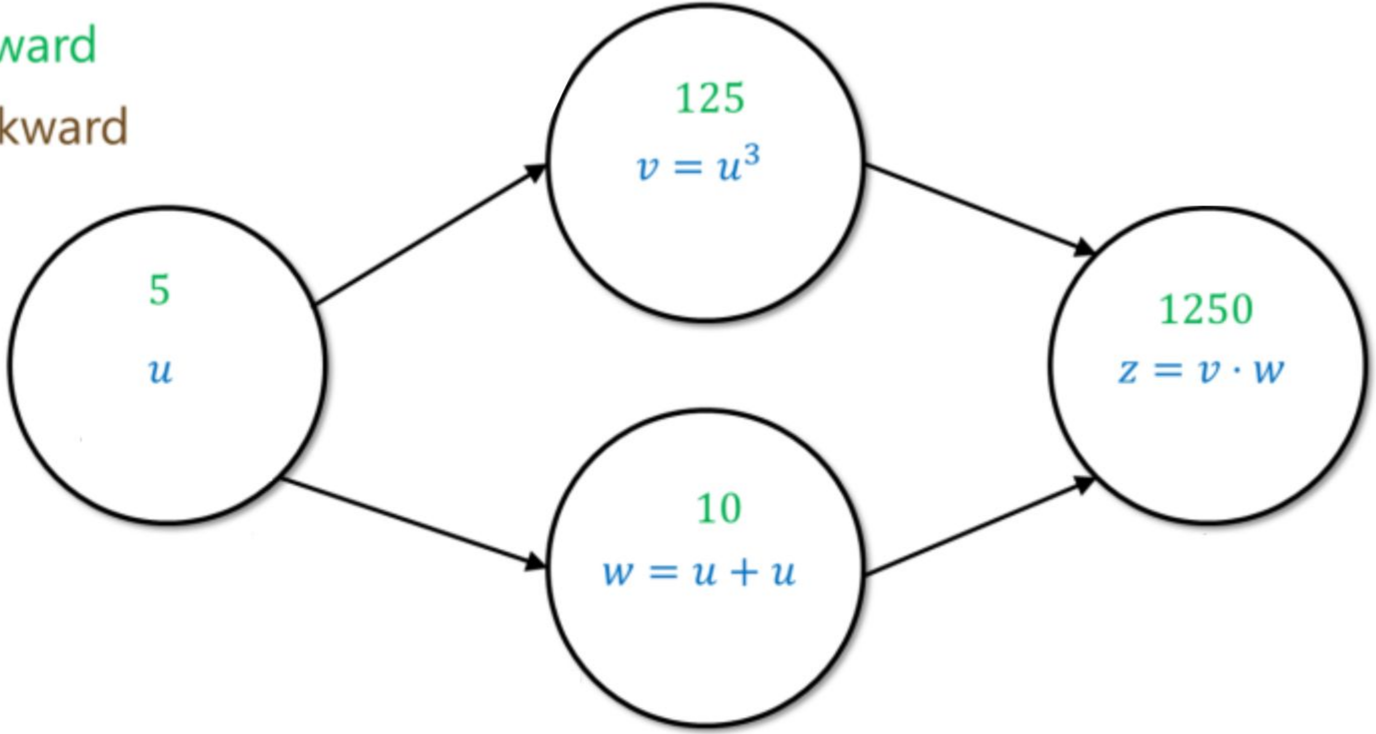
https://windowsontheory.org/2020/11/03/yet-another-backpropagation-tutorial/

Backpropagation- An Example

$$\frac{\partial z}{\partial u} = \left(\frac{\partial v}{\partial u} \cdot \frac{\partial z}{\partial v} + \frac{\partial w}{\partial u} \cdot \frac{\partial z}{\partial w}\right)$$

Forward

Backward



https://windowsontheory.org/2020/11/03/yet-another-backpropagation-tutorial/

Backpropagation- An Example

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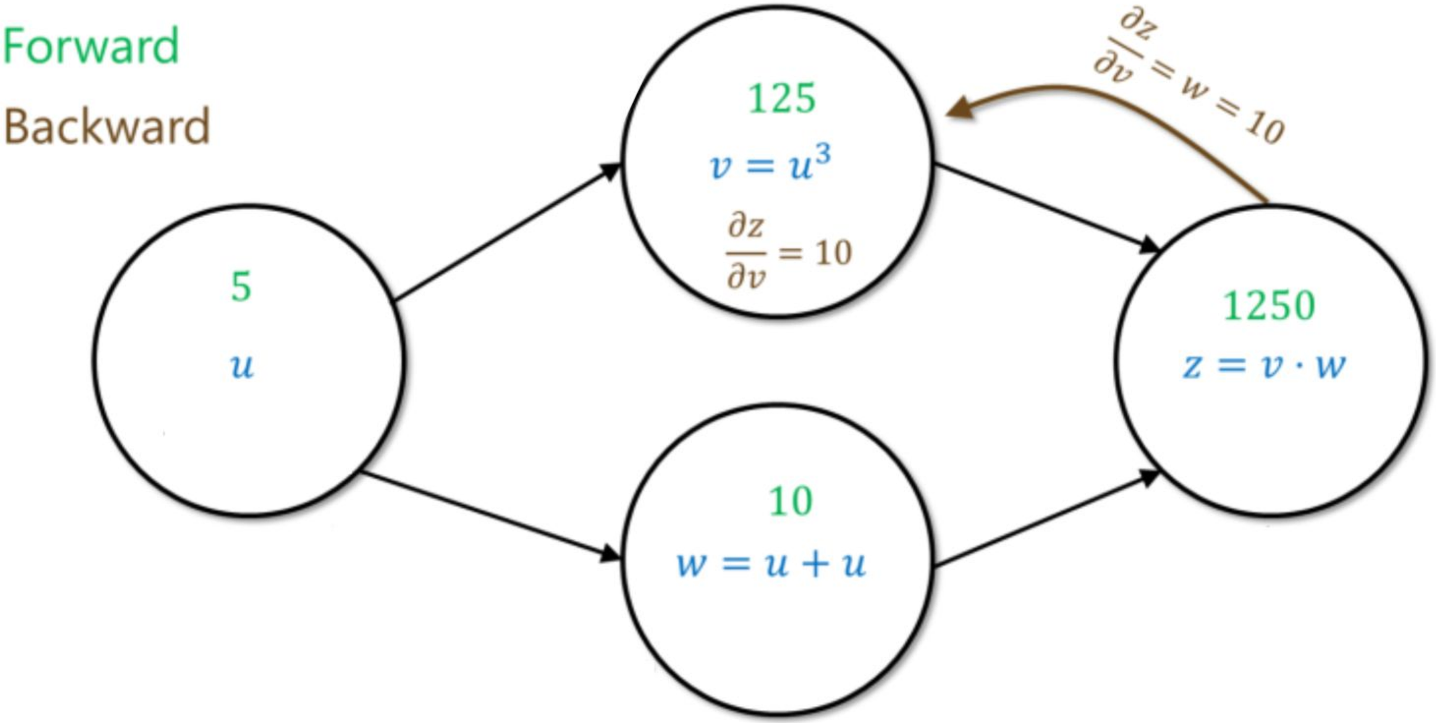

https://windowsontheory.org/2020/11/03/yet-another-backpropagation-tutorial/

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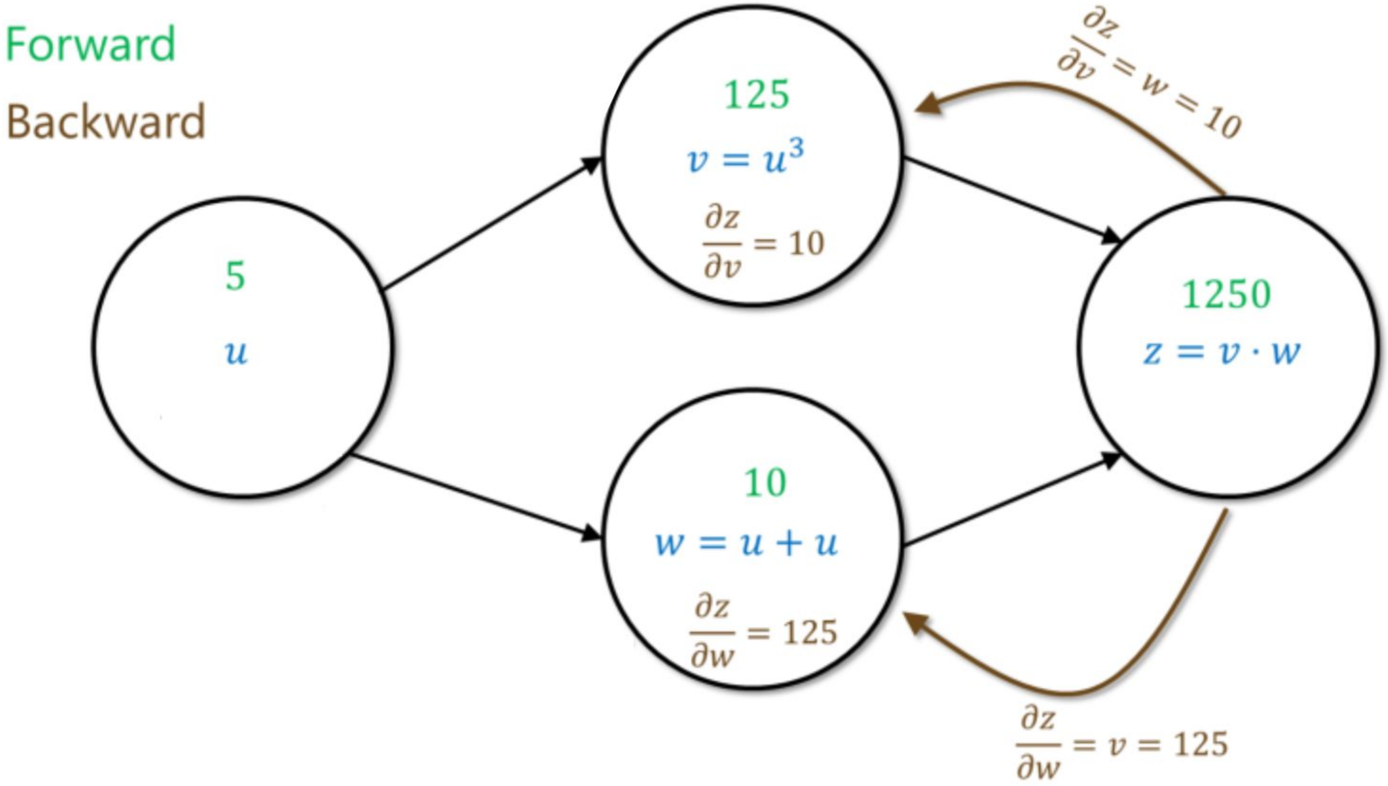
Backpropagation- An Example

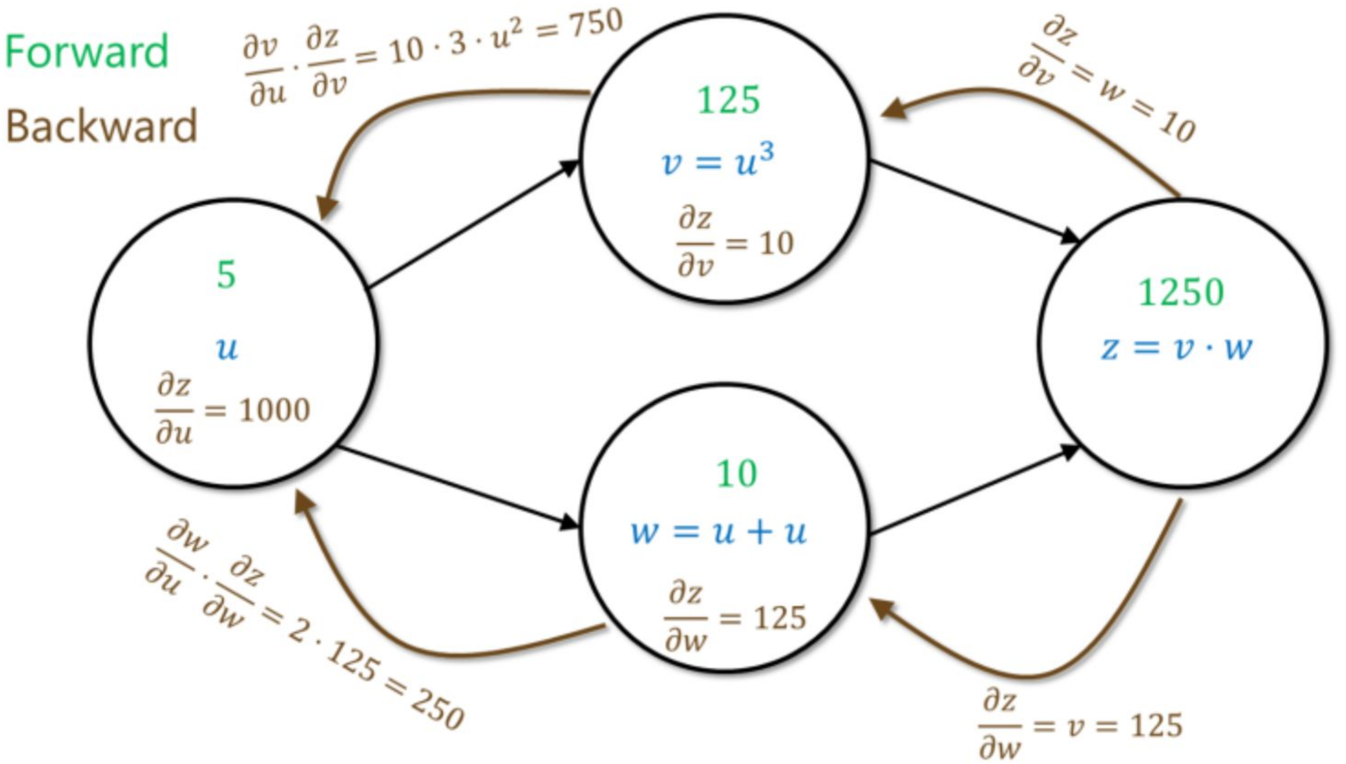
$$\frac{\partial z}{\partial u} = \left(\frac{\partial v}{\partial u} \cdot \frac{\partial z}{\partial v} + \frac{\partial w}{\partial u} \cdot \frac{\partial z}{\partial w}\right)$$



https://windowsontheory.org/2020/11/03/yet-another-backpropagation-tutorial/

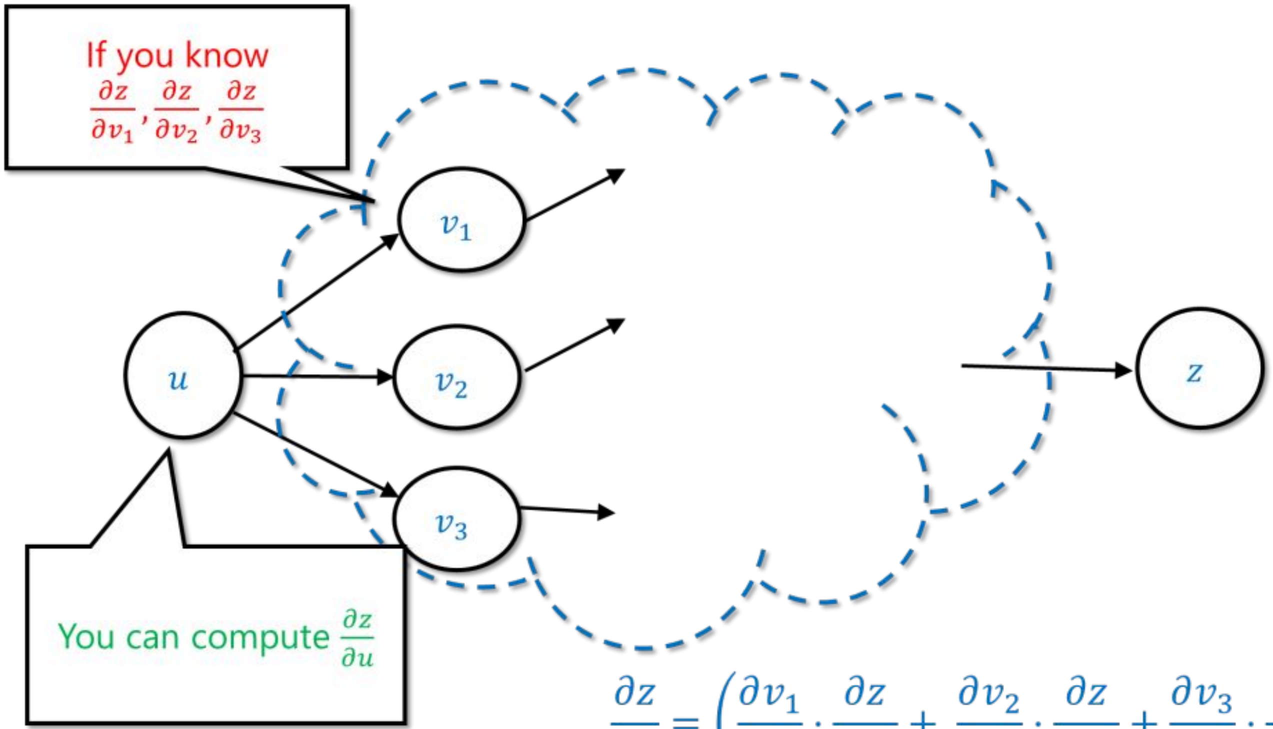
Backpropagation- An Example

$$\frac{\partial z}{\partial u} = \left(\frac{\partial v}{\partial u} \cdot \frac{\partial z}{\partial v} + \frac{\partial w}{\partial u} \cdot \frac{\partial z}{\partial w}\right)$$



https://windowsontheory.org/2020/11/03/yet-another-backpropagation-tutorial/

Backpropagation- Key Idea

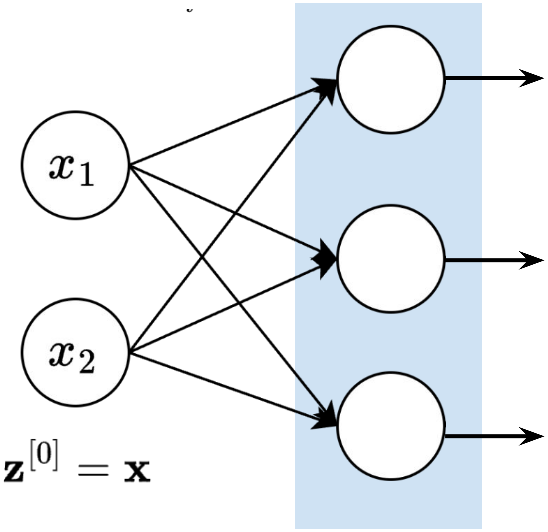


$$\frac{\partial z}{\partial u} = \left(\frac{\partial v_1}{\partial u} \cdot \frac{\partial z}{\partial v_1} + \frac{\partial v_2}{\partial u} \cdot \frac{\partial z}{\partial v_2} + \frac{\partial v_3}{\partial u} \cdot \frac{\partial z}{\partial v_3} \right)$$

https://windowsontheory.org/2020/11/03/yet-another-backpropagation-tutorial/

Forward Pass - MLP

$$\mathbf{a}^{[1]} = \mathbf{W}^{[1]}\mathbf{z}^{[0]}$$



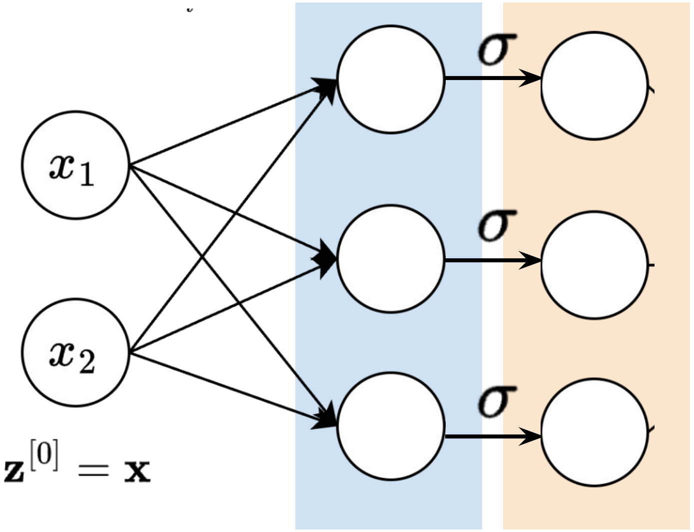
$$\mathbf{z}^{[0]} = \mathbf{x}$$

Algorithm Forward Pass through MLP

Input: input $\mathbf{x}$, weight matrices $\mathbf{W}^{[1]}, \ldots, \mathbf{W}^{[L]}$, bias vectors $\mathbf{b}^{[1]}, \ldots, \mathbf{b}^{[L]}$
$\mathbf{z}^{[0]} = \mathbf{x}$ ▷ Initialize input
for $l = 1$ **to** L **do**
  $\mathbf{a}^{[l]} = \mathbf{W}^{[l]}\mathbf{z}^{[l-1]} + \mathbf{b}^{[l]}$ ▷ Linear transformation
  $\mathbf{z}^{[l]} = \sigma^{[l]}(\mathbf{a}^{[l]})$ ▷ Nonlinear activation
end for
Output: $\mathbf{z}^{[L]}$

Forward Pass - MLP

$$\mathbf{a}^{[1]} = \mathbf{W}^{[1]}\mathbf{z}^{[0]}$$



$$\mathbf{z}^{[0]} = \mathbf{x}$$

$$\mathbf{z}^{[1]} = \sigma(\mathbf{a}^{[1]})$$

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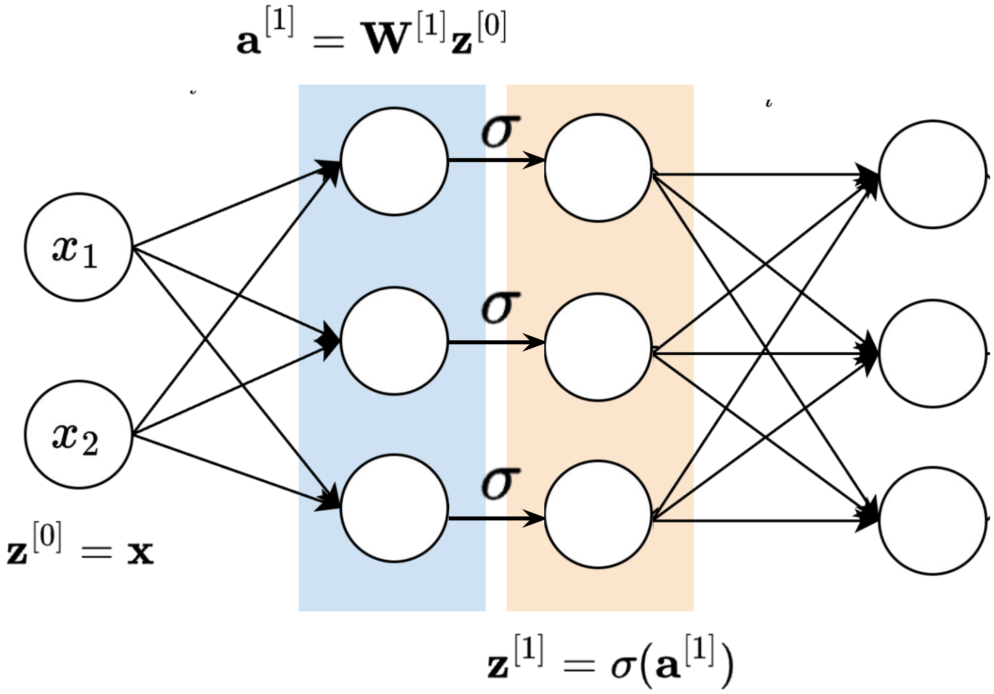
Forward Pass - MLP



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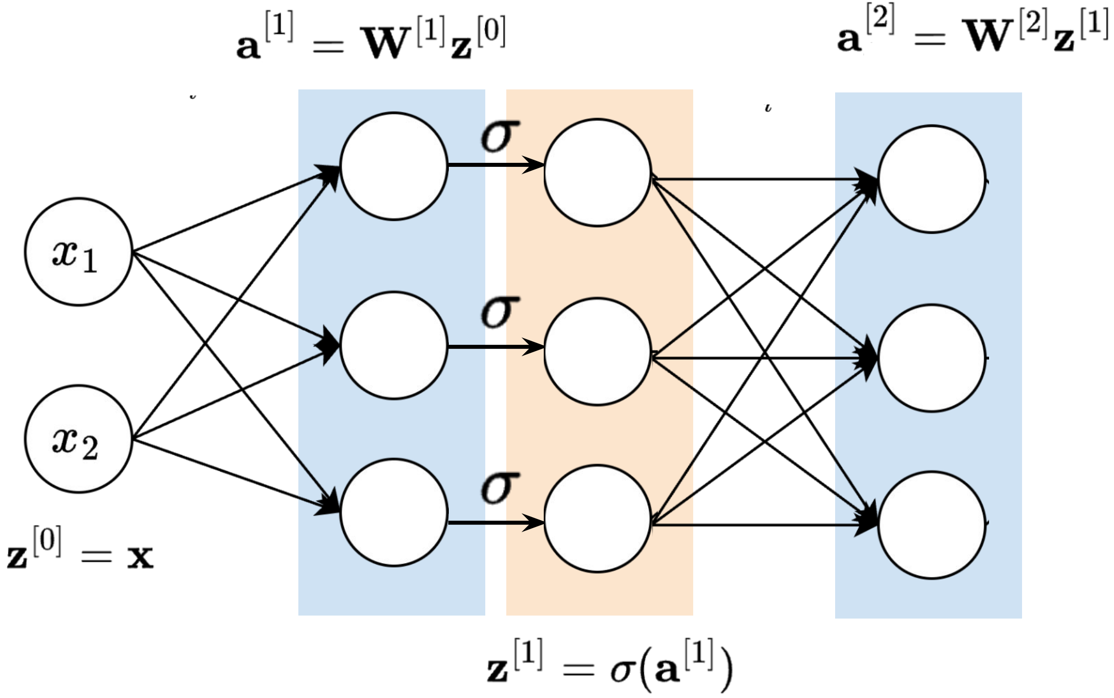
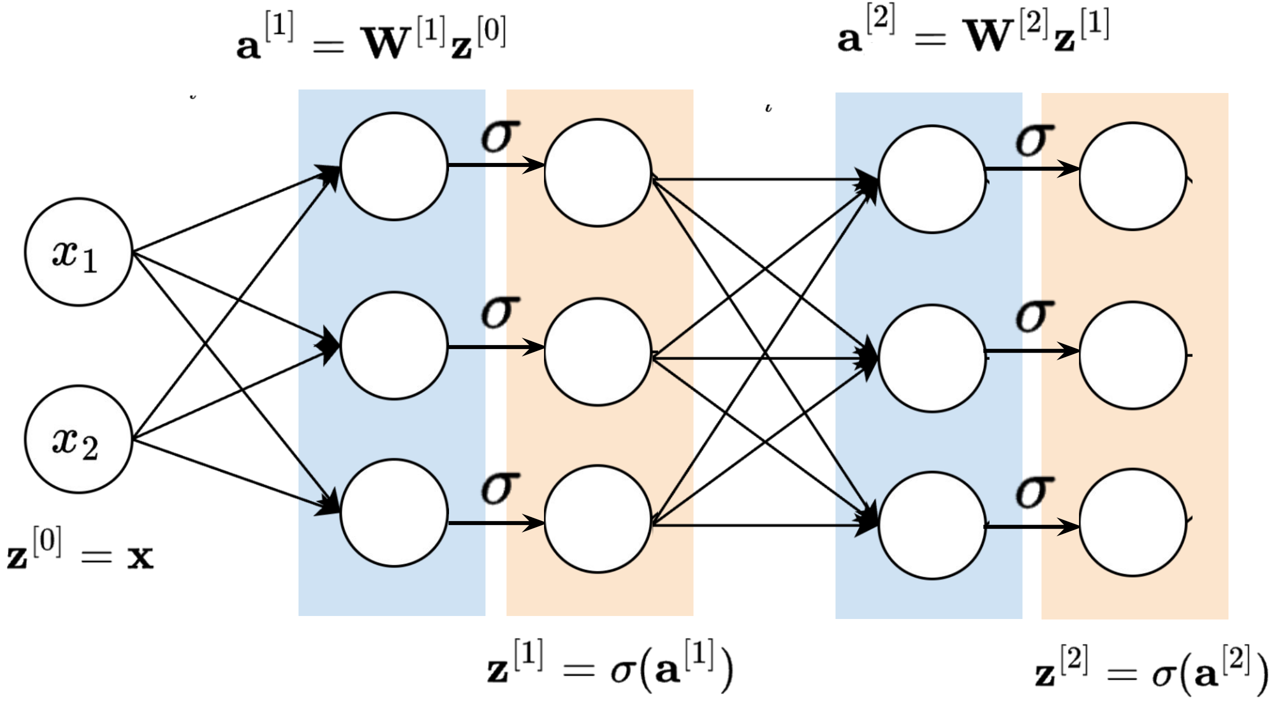
Forward Pass - MLP



Forward Pass - MLP



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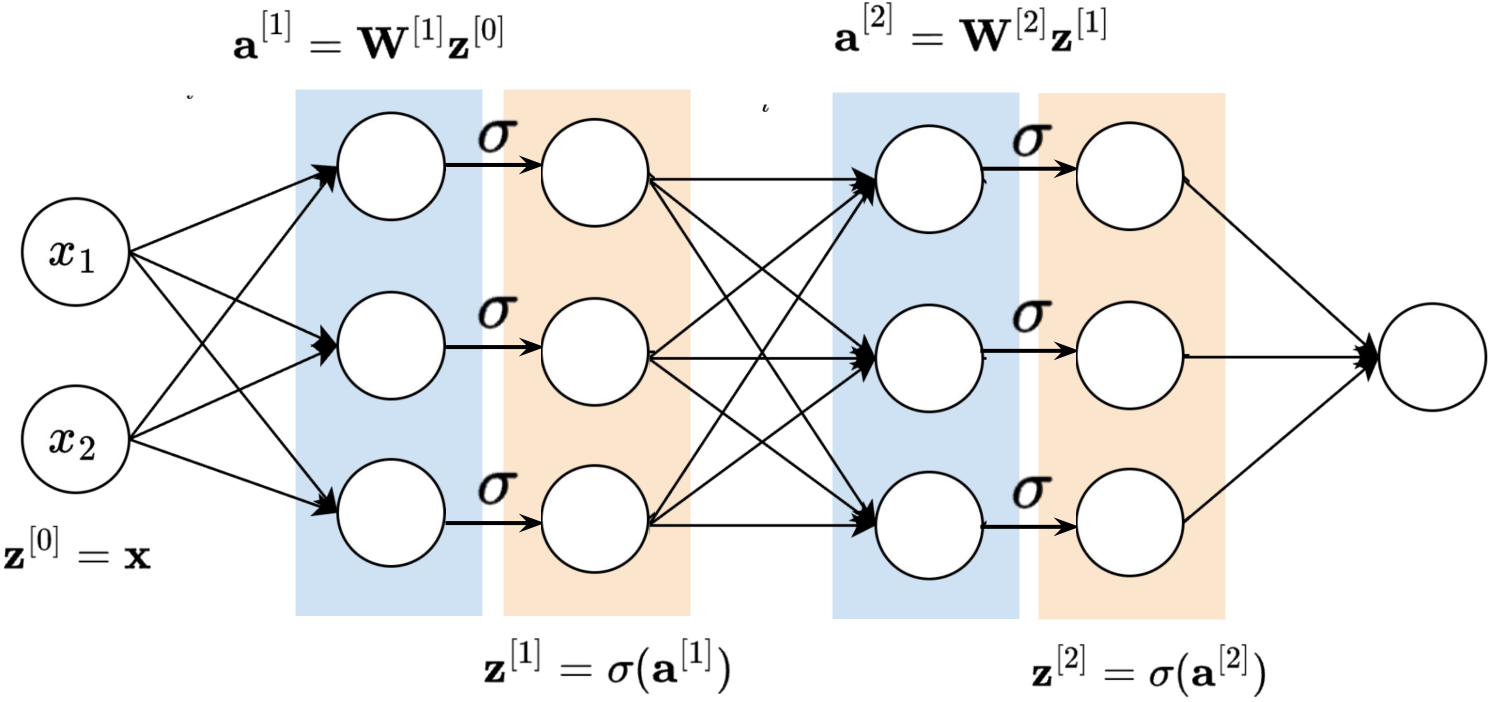
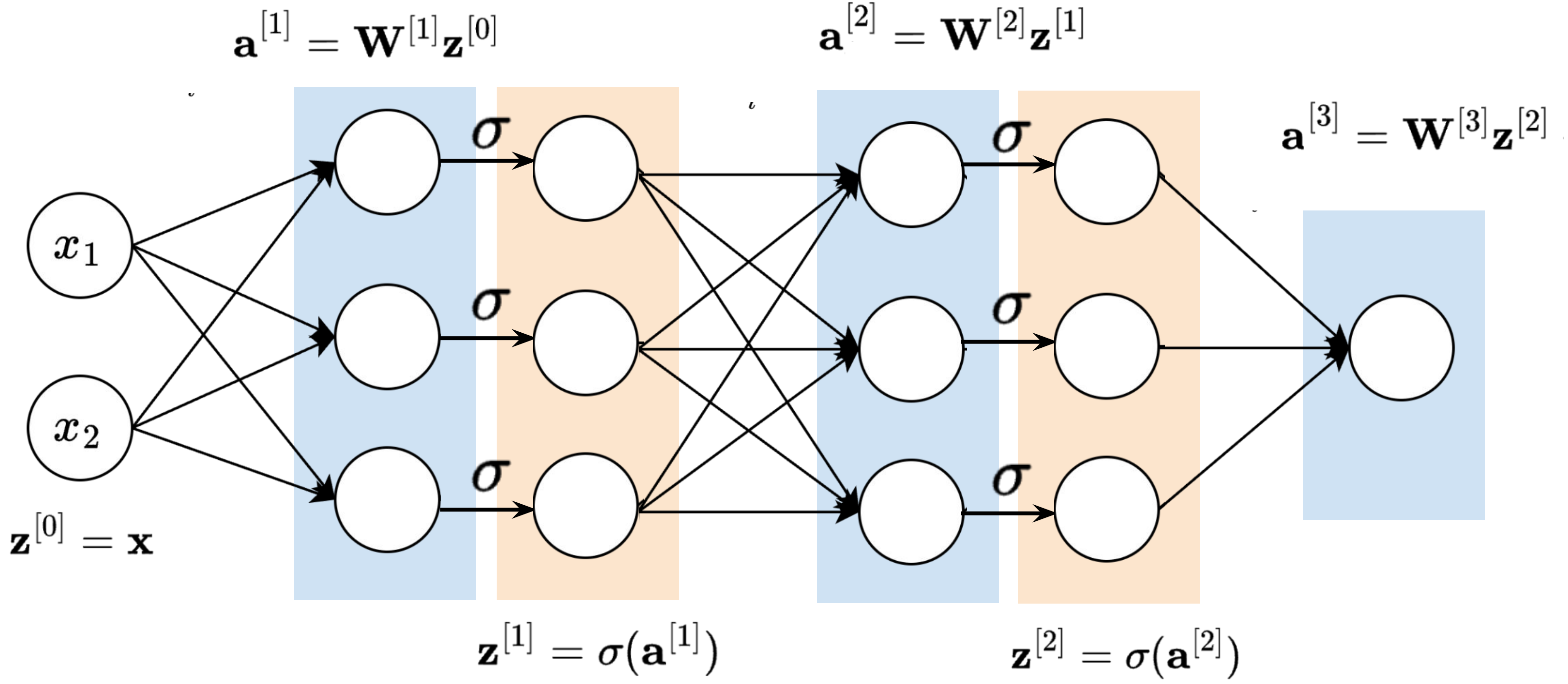
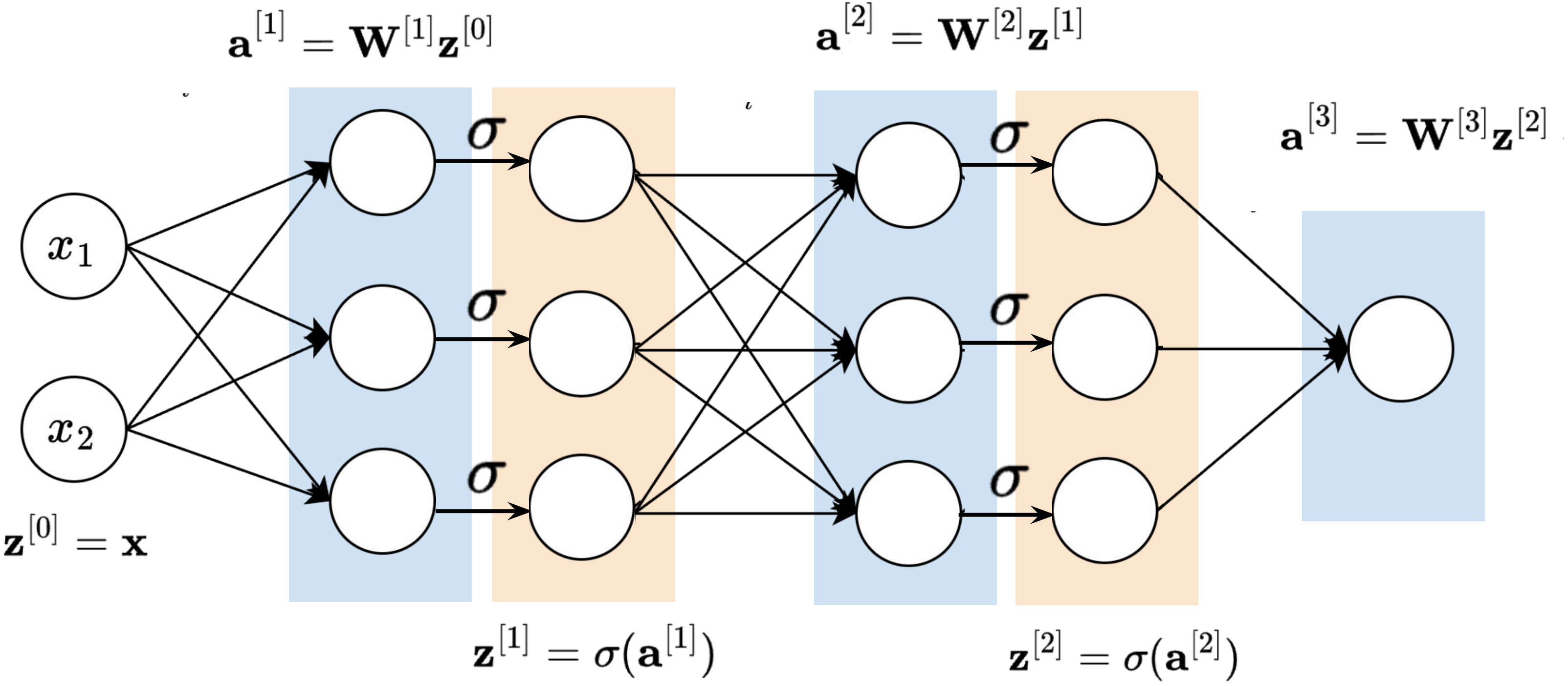
Forward Pass - MLP



Forward Pass - MLP



Forward Pass - MLP



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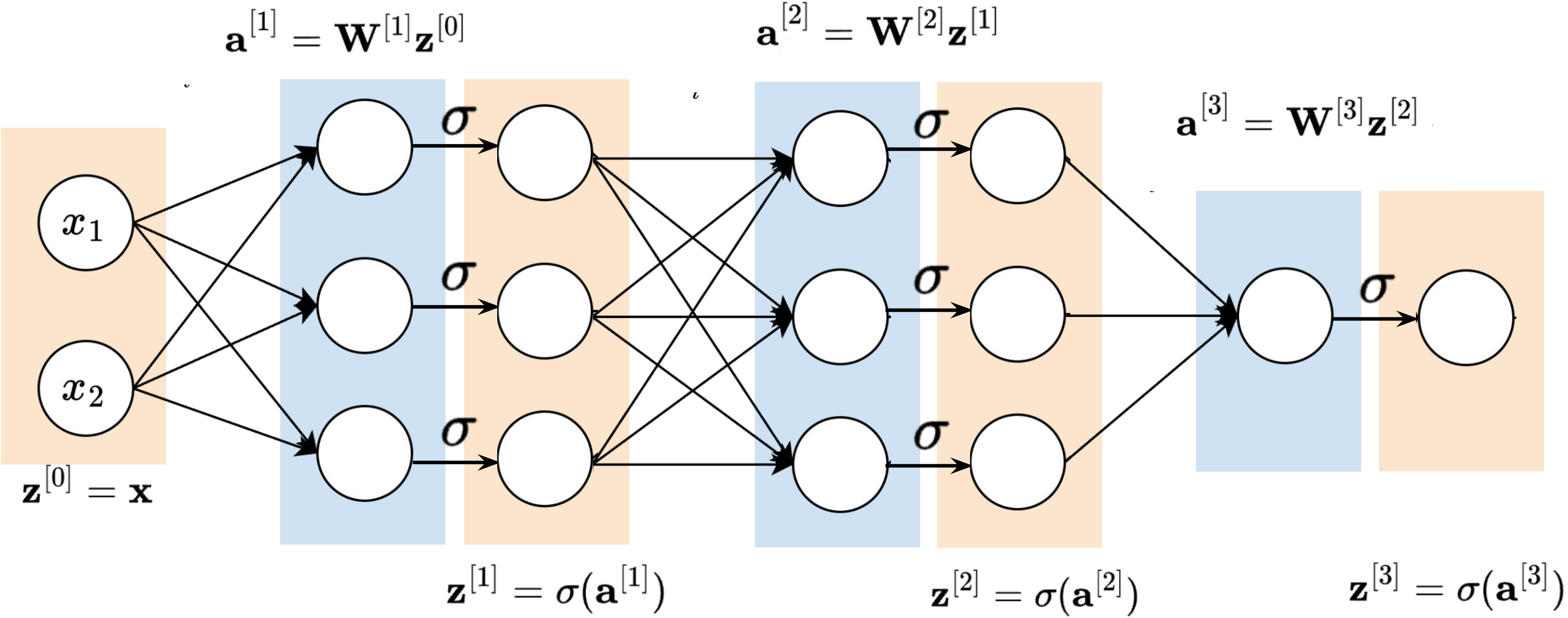
Forward Pass - MLP



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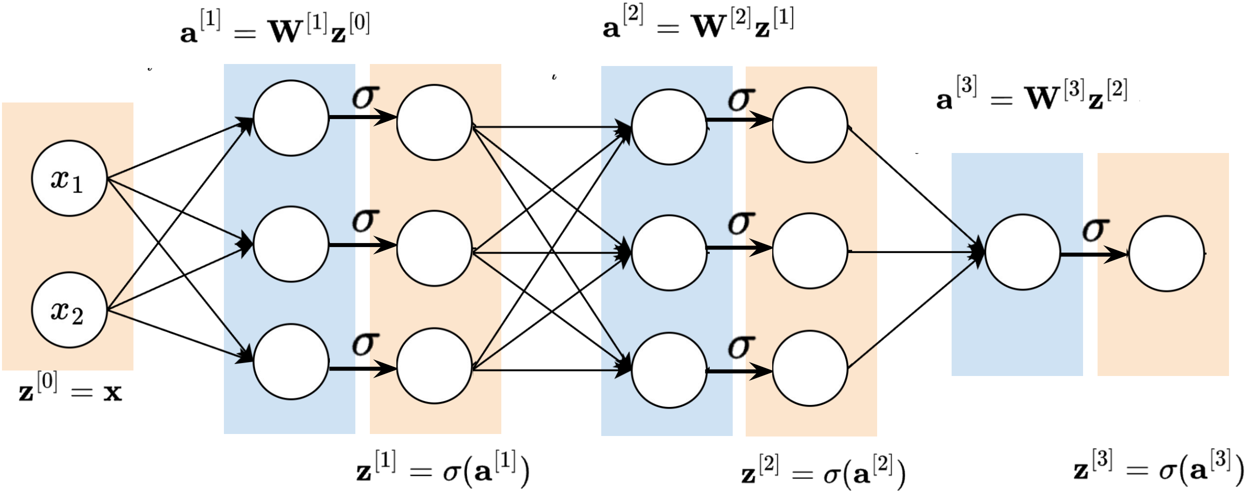
Forward Pass - MLP

Algorithm Forward Pass through MLP

1: **Input:** input $\mathbf{x}$, weight matrices $\mathbf{W}^{[1]}, \ldots, \mathbf{W}^{[L]}$, bias vectors $\mathbf{b}^{[1]}, \ldots, \mathbf{b}^{[L]}$
2: $\mathbf{z}^{[0]} = \mathbf{x}$ ▷ Initialize input
3: **for** $l = 1$ **to** L **do**
4: $\quad \mathbf{a}^{[l]} = \mathbf{W}^{[l]}\mathbf{z}^{[l-1]} + \mathbf{b}^{[l]}$ ▷ Linear transformation
5: $\quad \mathbf{z}^{[l]} = \sigma^{[l]}(\mathbf{a}^{[l]})$ ▷ Nonlinear activation
6: **end for**
7: **Output:** $\mathbf{z}^{[L]}$



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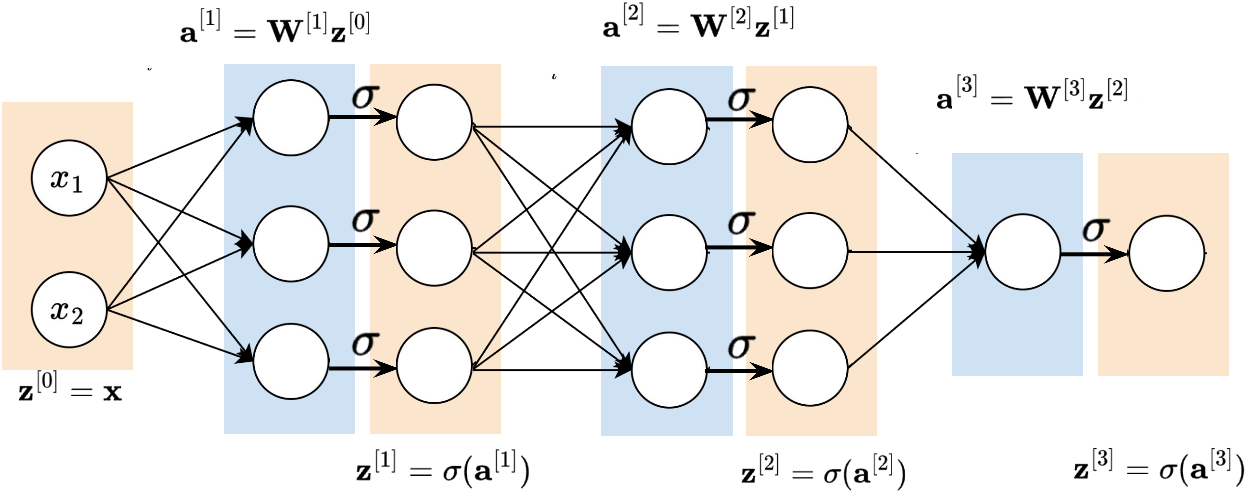
Forward Pass - MLP

Algorithm Forward Pass through MLP

```
Input: input x, weight matrices W^[1], ..., W^[L], bias vectors b^[1], ..., b^[L]
z^[0] = x                                      ▷ Initialize input
for l = 1 to L do
    a^[l] = W^[l] z^[l-1] + b^[l]              ▷ Linear transformation
    z^[l] = σ^[l](a^[l])                       ▷ Nonlinear activation
end for
Output: z^[L]
```



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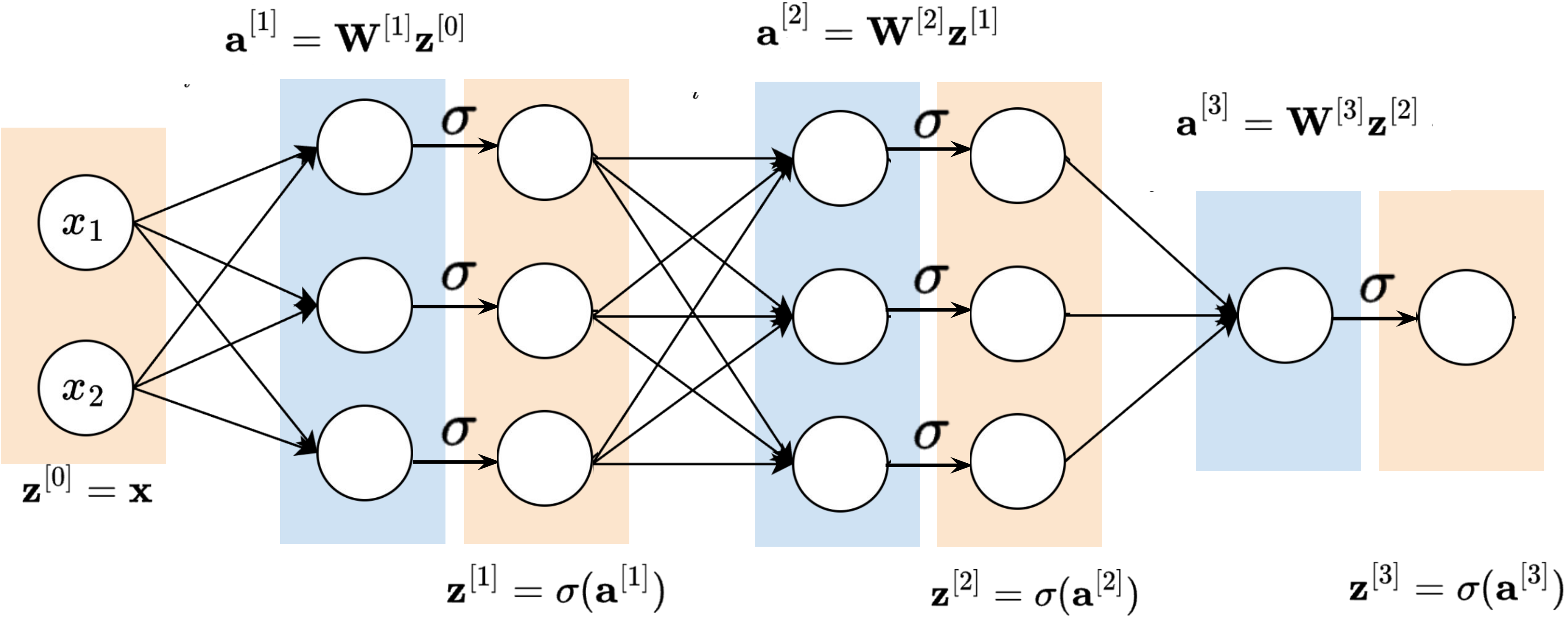
Backprop

$\text{Loss} = \mathcal{L}(\mathbf{z}^{[3]}, \mathbf{y})$



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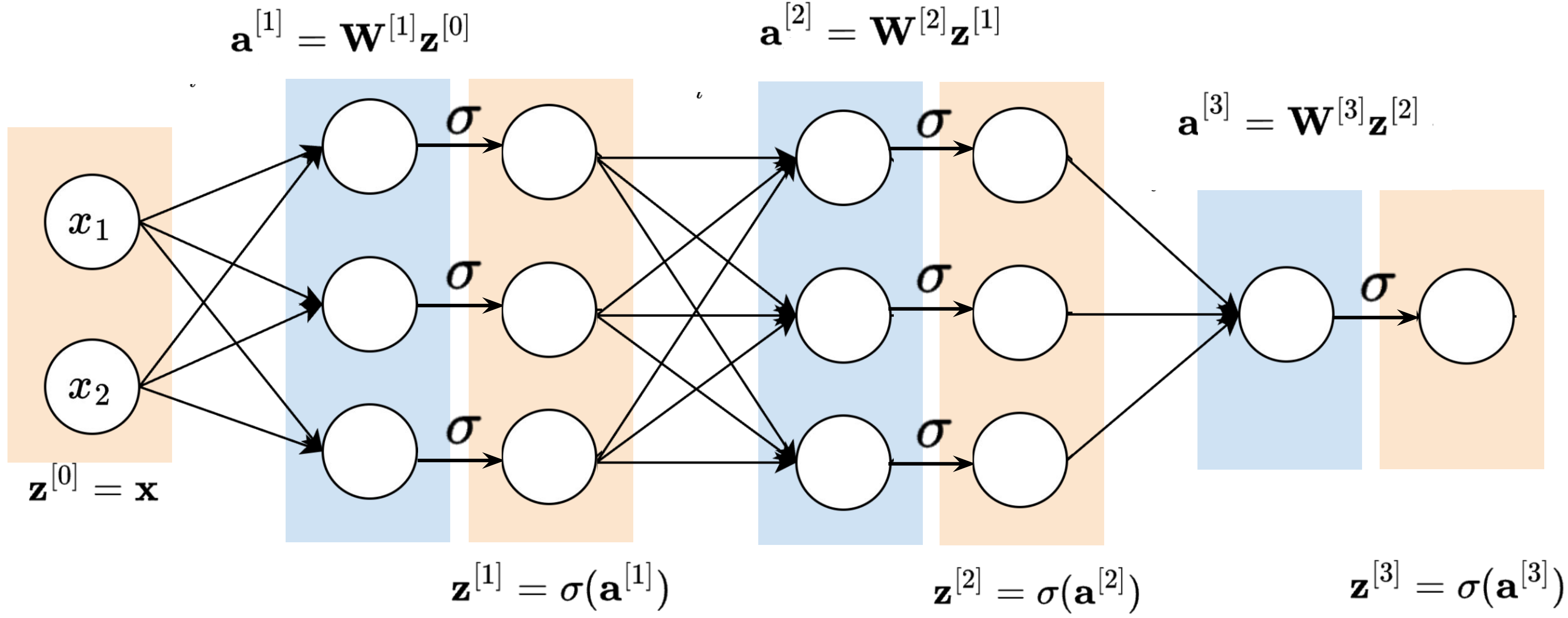
Backprop

$\text{Loss} = \mathcal{L}(\mathbf{z}^{[3]}, \mathbf{y})$

We can directly compute $\frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}}$!



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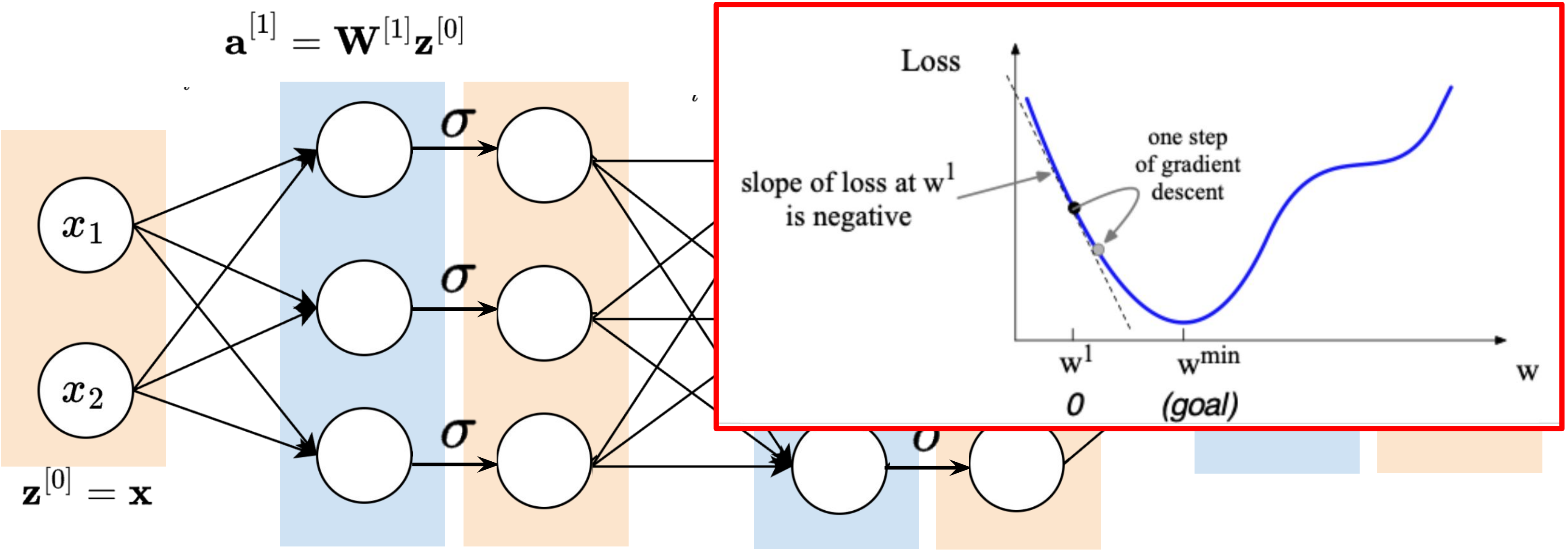
Backprop

$\text{Loss} = \mathcal{L}(\mathbf{z}^{[3]}, \mathbf{y})$

We can directly compute $\frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}}$!

$\mathbf{a}^{[1]} = \mathbf{W}^{[1]}\mathbf{z}^{[0]}$



$\mathbf{z}^{[0]} = \mathbf{x}$

$\mathbf{z}^{[1]} = \sigma(\mathbf{a}^{[1]})$

$\mathbf{z}^{[2]} = \sigma(\mathbf{a}^{[2]})$

$\mathbf{z}^{[3]} = \sigma(\mathbf{a}^{[3]})$

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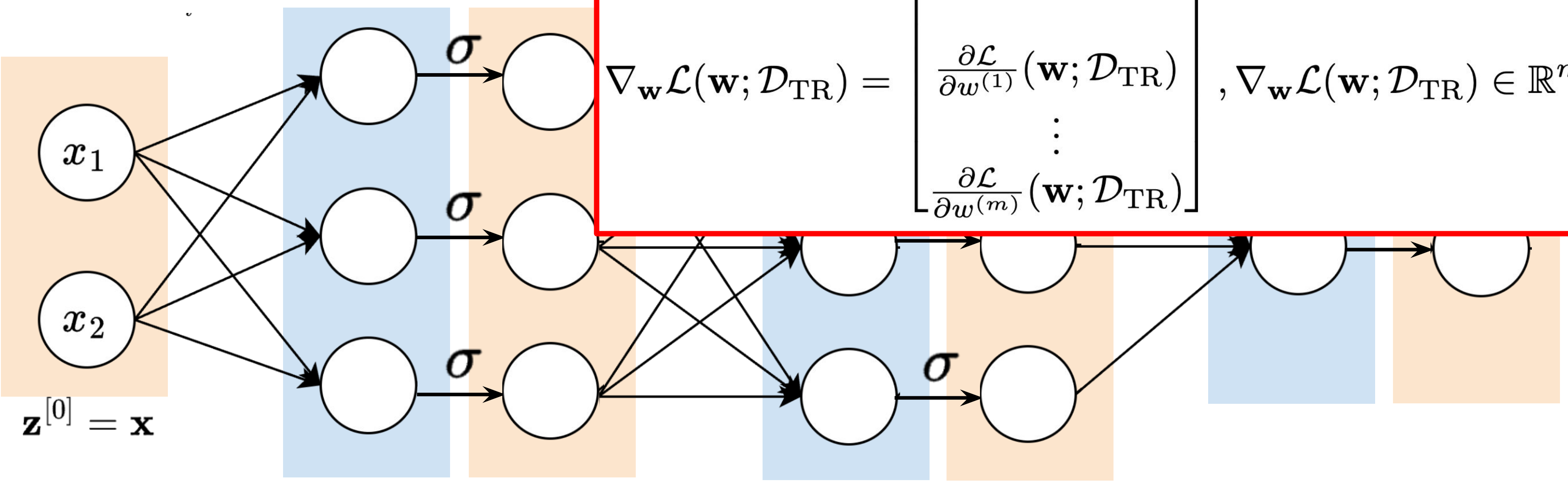
Backprop

Loss $= \mathcal{L}(\mathbf{z}^{[3]}, \mathbf{y})$

We can directly compute $\frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}}$!

$$\nabla_{\mathbf{w}} \mathcal{L}(\mathbf{w}; \mathcal{D}_{\text{TR}}) = \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial w^{(0)}}(\mathbf{w}; \mathcal{D}_{\text{TR}}) \\ \frac{\partial \mathcal{L}}{\partial w^{(1)}}(\mathbf{w}; \mathcal{D}_{\text{TR}}) \\ \vdots \\ \frac{\partial \mathcal{L}}{\partial w^{(m)}}(\mathbf{w}; \mathcal{D}_{\text{TR}}) \end{bmatrix}, \nabla_{\mathbf{w}} \mathcal{L}(\mathbf{w}; \mathcal{D}_{\text{TR}}) \in \mathbb{R}^m$$

$\mathbf{a}^{[1]} = \mathbf{W}^{[1]}\mathbf{z}^{[0]}$



$\mathbf{z}^{[0]} = \mathbf{x}$

$\mathbf{z}^{[1]} = \sigma(\mathbf{a}^{[1]})$

$\mathbf{z}^{[2]} = \sigma(\mathbf{a}^{[2]})$

$\mathbf{z}^{[3]} = \sigma(\mathbf{a}^{[3]})$

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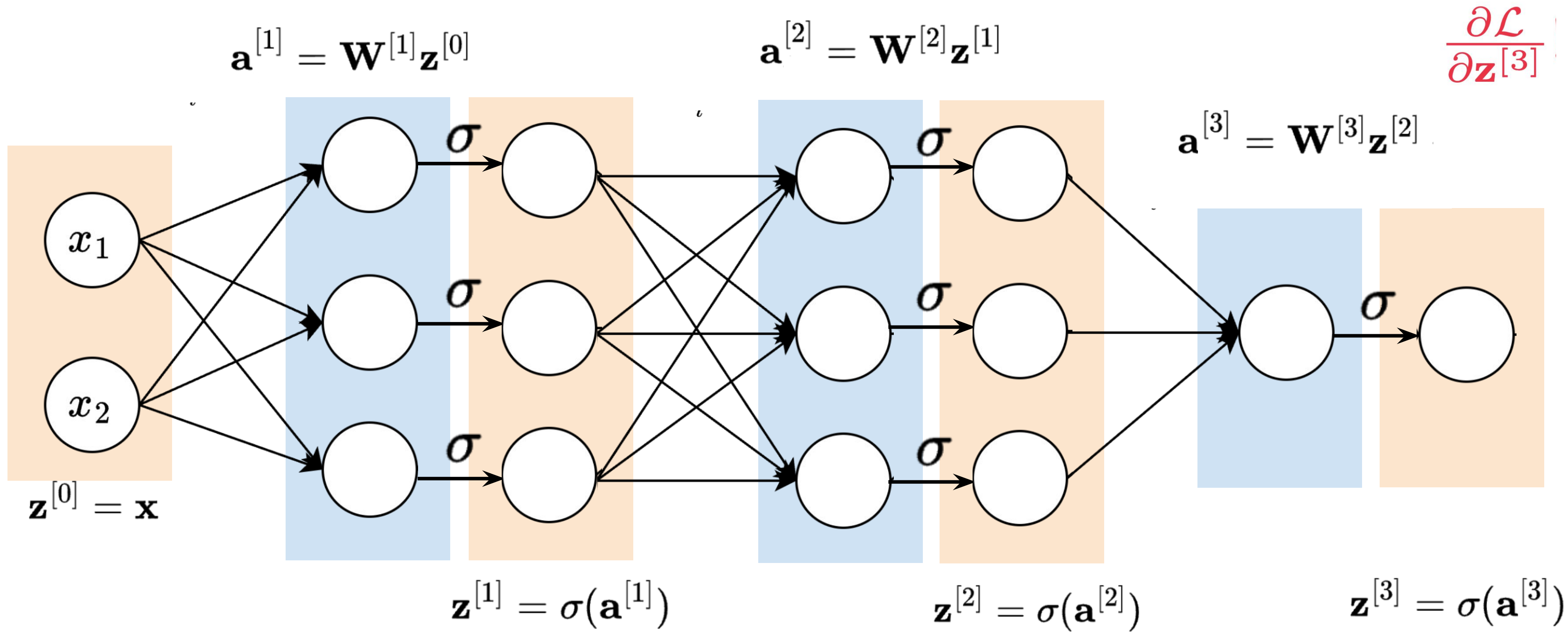
Backprop

Loss $= \mathcal{L}(\mathbf{z}^{[3]}, \mathbf{y})$



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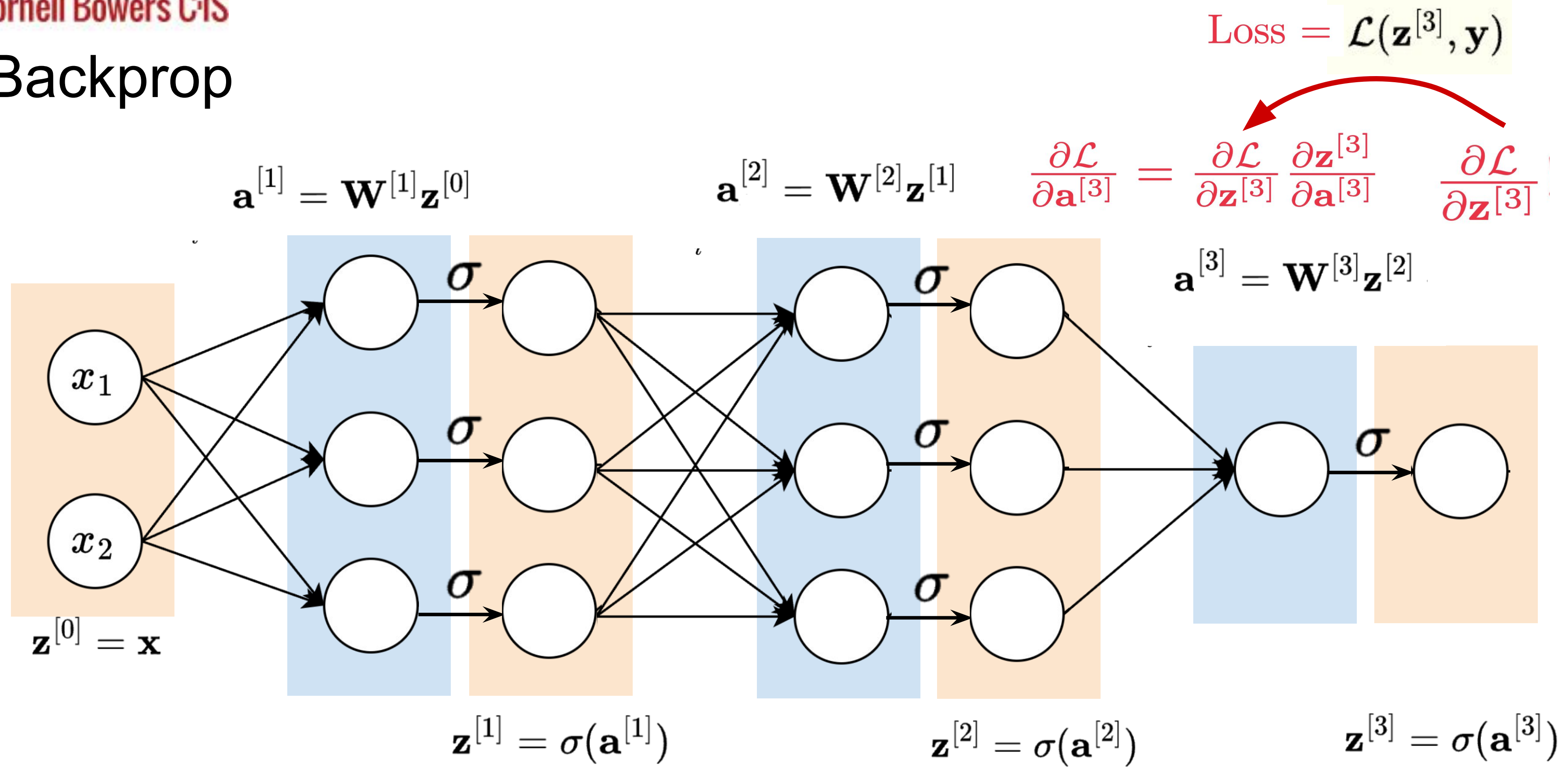
Backprop



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Backprop

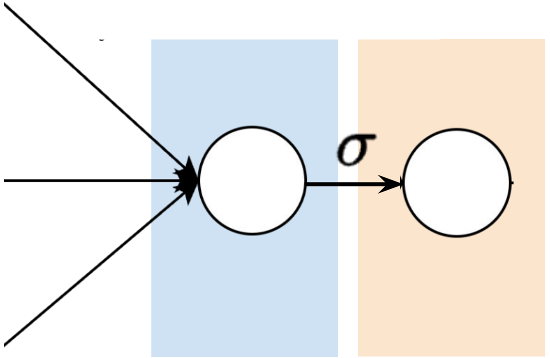
$$\text{Loss} = \mathcal{L}(\mathbf{z}^{[3]}, \mathbf{y})$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[3]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}} \frac{\partial \mathbf{z}^{[3]}}{\partial \mathbf{a}^{[3]}}$$

$$= \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}} \odot \sigma^{[3]\prime}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}}$$

$$\mathbf{a}^{[3]} = \mathbf{W}^{[3]}\mathbf{z}^{[2]}$$



$$\mathbf{z}^{[3]} = \sigma(\mathbf{a}^{[3]})$$

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Backprop

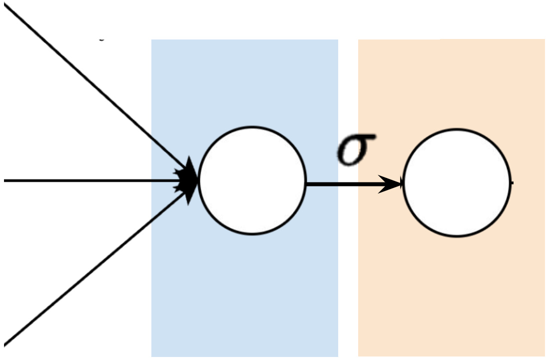
$$\text{Loss} = \mathcal{L}(\mathbf{z}^{[3]}, \mathbf{y})$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}}$$

$$\delta^{[3]} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[3]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}} \frac{\partial \mathbf{z}^{[3]}}{\partial \mathbf{a}^{[3]}}$$

$$= \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}} \odot \sigma^{[3]'}$$

$$\mathbf{a}^{[3]} = \mathbf{W}^{[3]}\mathbf{z}^{[2]}$$



$$\mathbf{z}^{[3]} = \sigma(\mathbf{a}^{[3]})$$

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Backprop

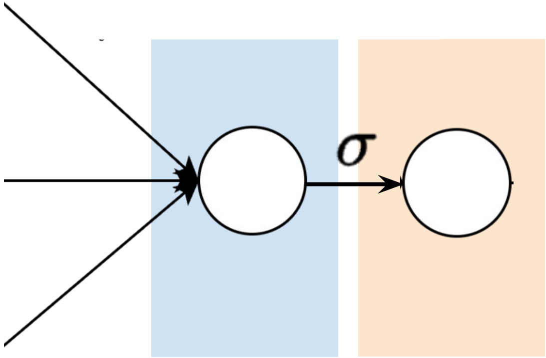
$$\text{Loss} = \mathcal{L}(\mathbf{z}^{[3]}, \mathbf{y})$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}}$$

$$\delta^{[3]} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[3]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}} \frac{\partial \mathbf{z}^{[3]}}{\partial \mathbf{a}^{[3]}}$$

$$= \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}} \odot \sigma^{[3]\prime}$$

$$\mathbf{a}^{[3]} = \mathbf{W}^{[3]}\mathbf{z}^{[2]}$$



$$\frac{\partial \mathcal{L}}{\partial \mathbf{W}^{[3]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[3]}} \frac{\partial \mathbf{a}^{[3]}}{\partial \mathbf{W}^{[3]}}$$

$$= \delta^{[3]} (\mathbf{z}^{[2]})^T$$

$$\mathbf{z}^{[3]} = \sigma(\mathbf{a}^{[3]})$$

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Backprop

$$\text{Loss} = \mathcal{L}(\mathbf{z}^{[3]}, \mathbf{y})$$

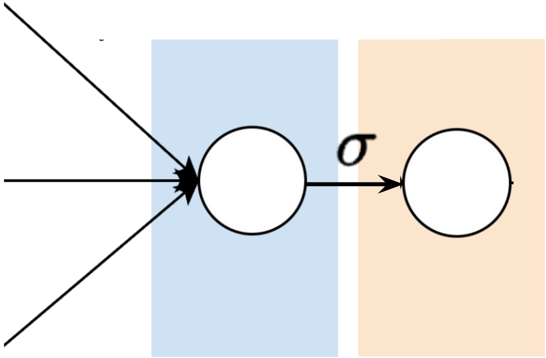
$$\frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}}$$

For propagation to next layer:

$$\boxed{\delta^{[3]} =} \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[3]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}} \frac{\partial \mathbf{z}^{[3]}}{\partial \mathbf{a}^{[3]}}$$

$$= \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}} \odot \sigma^{[3]\prime}$$

$$\mathbf{a}^{[3]} = \mathbf{W}^{[3]}\mathbf{z}^{[2]}$$



For weight updates:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{W}^{[3]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[3]}} \frac{\partial \mathbf{a}^{[3]}}{\partial \mathbf{W}^{[3]}}$$

$$= \delta^{[3]} (\mathbf{z}^{[2]})^T$$

$$\mathbf{z}^{[3]} = \sigma(\mathbf{a}^{[3]})$$

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Backprop

$$\text{Loss} = \mathcal{L}(\mathbf{z}^{[3]}, \mathbf{y})$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}}$$

$$\mathbf{a}^{[3]} = \mathbf{W}^{[3]}\mathbf{z}^{[2]}$$

For propagation to next layer:

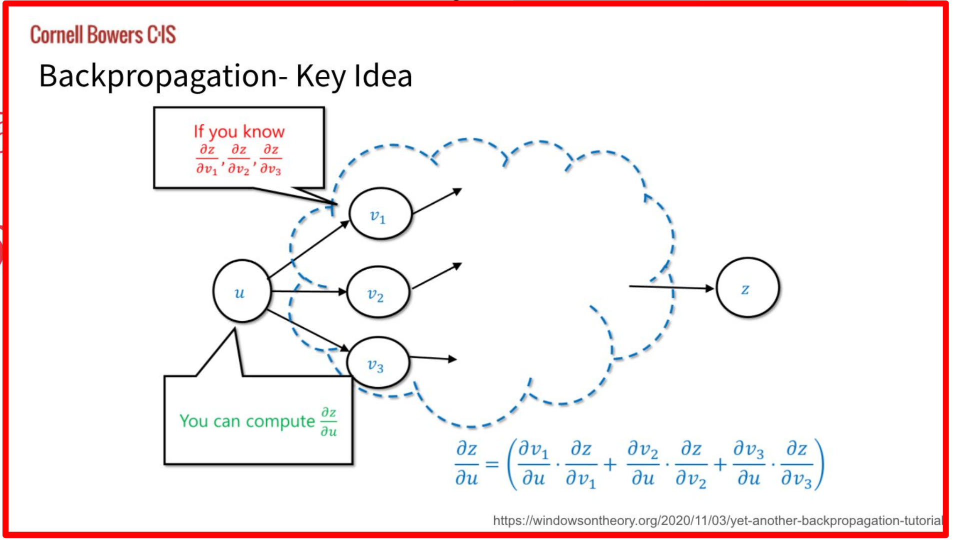
$$\delta^{[3]} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[3]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}} \frac{\partial \mathbf{z}^{[3]}}{\partial \mathbf{a}^{[3]}}$$

$$= \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}} \odot \sigma^{[3]'}$$

For weight updates:

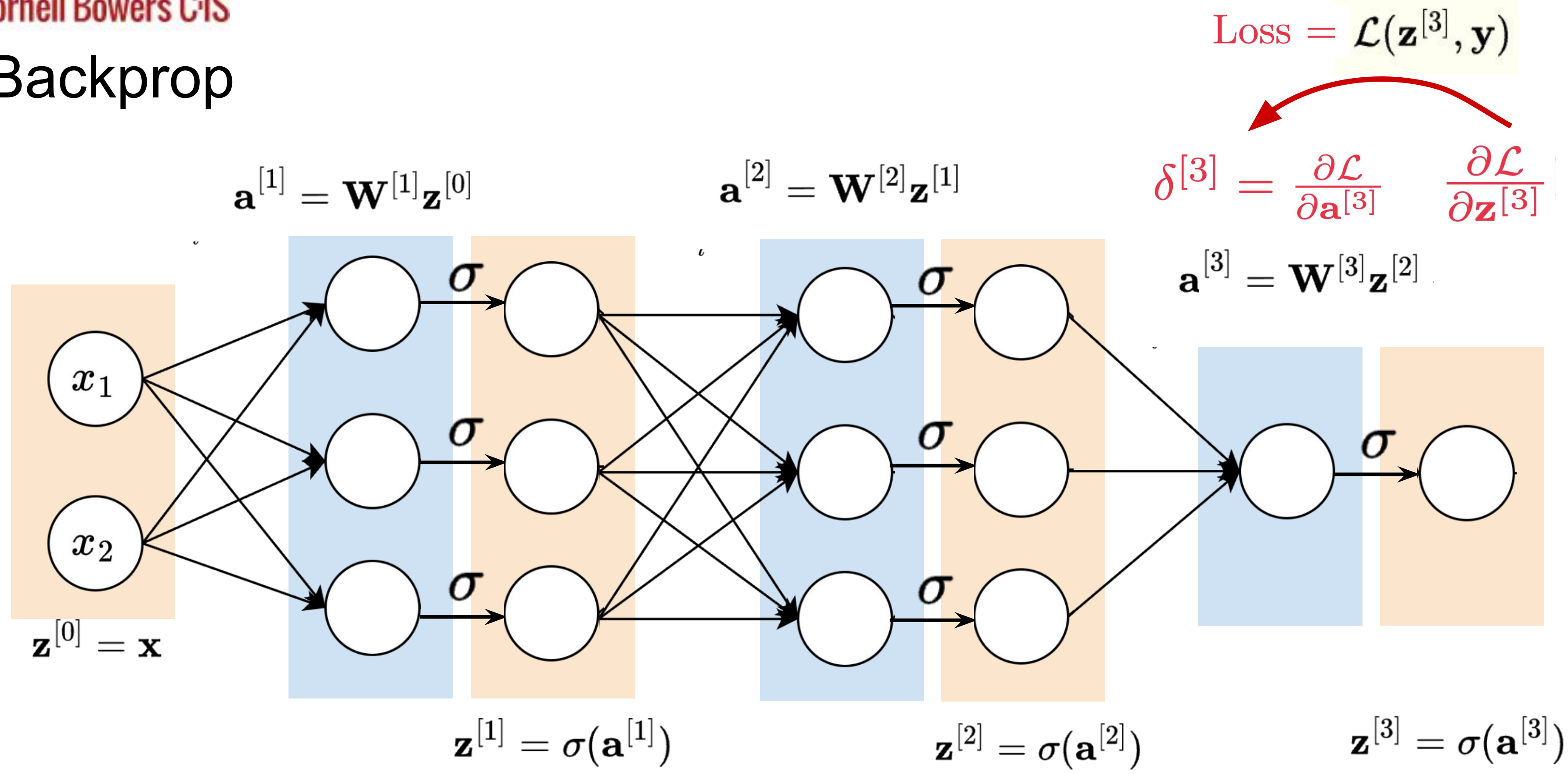
$$\frac{\partial \mathcal{L}}{\partial \mathbf{W}^{[3]}} = \partial \ldots$$

$$= \delta \ldots$$



Cornell Bowers C·IS

Backprop

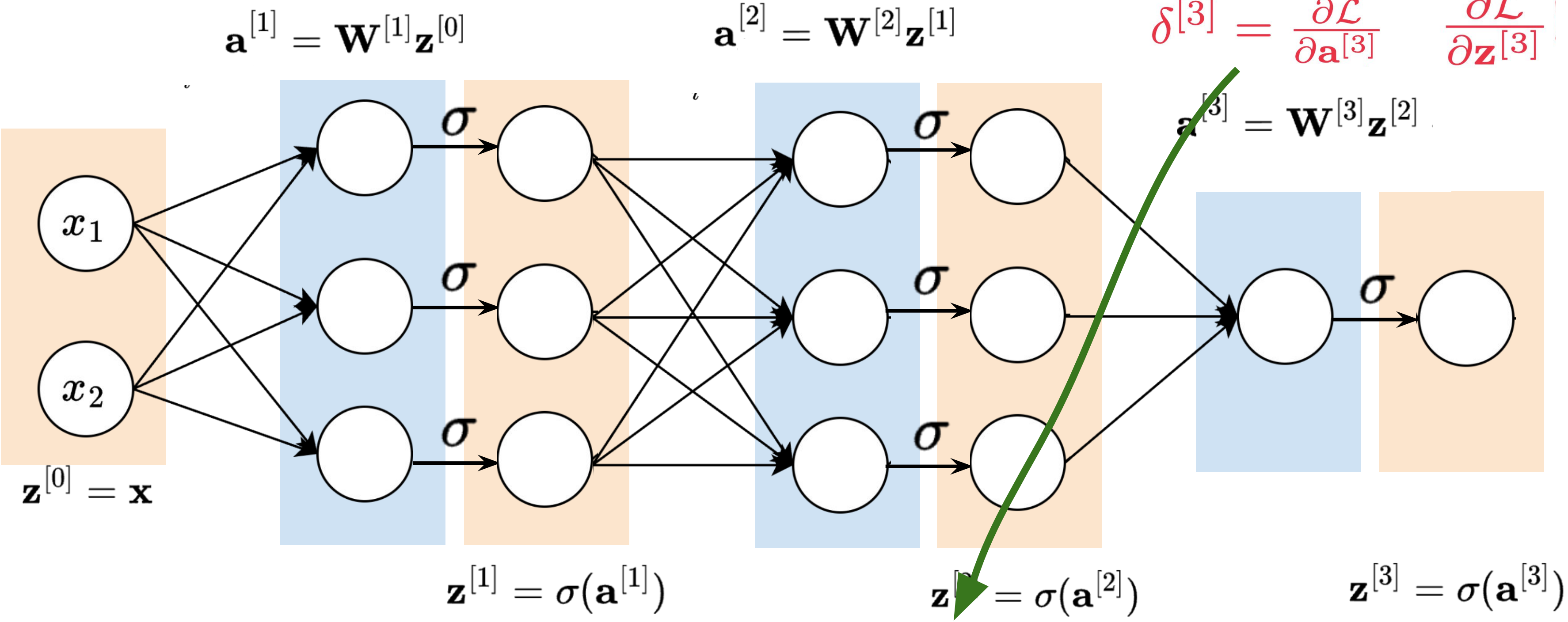


Cornell Bowers CIS

Backprop

$\text{Loss} = \mathcal{L}(\mathbf{z}^{[3]}, \mathbf{y})$

$\delta^{[3]} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[3]}} \quad \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}}$



$$\frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[2]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[3]}} \frac{\partial \mathbf{a}^{[3]}}{\partial \mathbf{z}^{[2]}} = (W^{[3]})^T \delta^{[3]}$$

Cornell Bowers C·IS

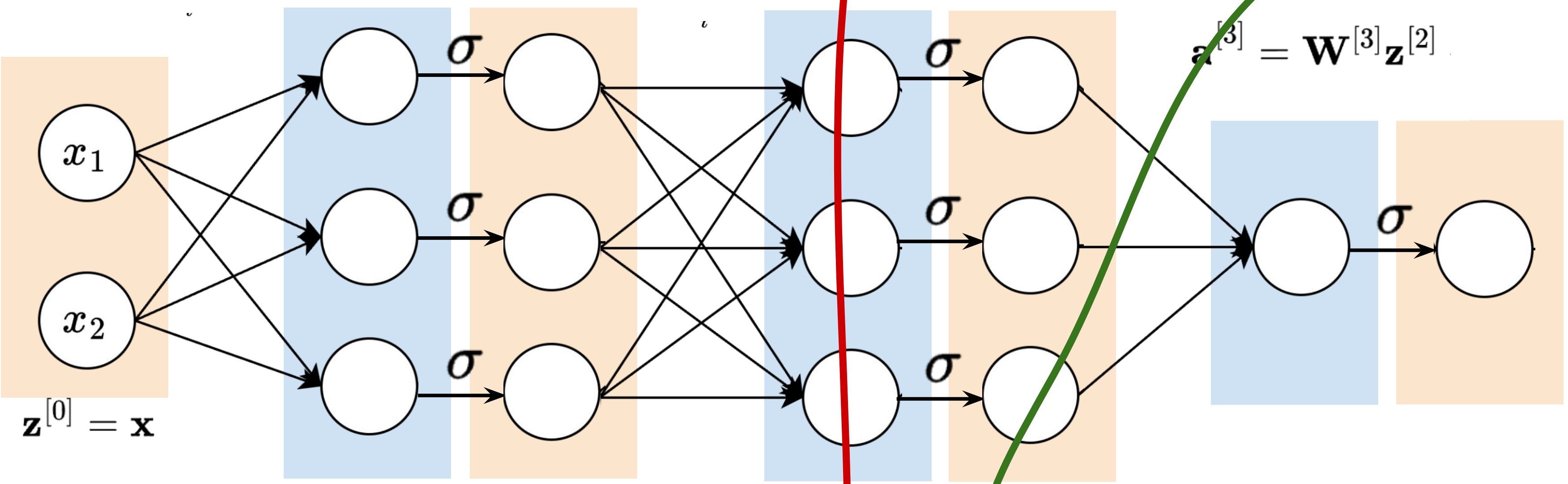
Backprop

$$\delta^{[2]} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[2]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[2]}} \frac{\partial \mathbf{z}^{[2]}}{\partial \mathbf{a}^{[2]}}$$

$$\text{Loss} = \mathcal{L}(\mathbf{z}^{[3]}, \mathbf{y})$$

$$\delta^{[3]} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[3]}} \qquad \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}}$$

$$\mathbf{a}^{[1]} = \mathbf{W}^{[1]}\mathbf{z}^{[0]} \qquad \mathbf{a}^{[2]} = \mathbf{W}^{[2]}\mathbf{z}^{[1]} \qquad \mathbf{a}^{[3]} = \mathbf{W}^{[3]}\mathbf{z}^{[2]}$$



$$\mathbf{z}^{[0]} = \mathbf{x} \qquad \mathbf{z}^{[1]} = \sigma(\mathbf{a}^{[1]}) \qquad \mathbf{z}^{[2]} = \sigma(\mathbf{a}^{[2]}) \qquad \mathbf{z}^{[3]} = \sigma(\mathbf{a}^{[3]})$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[2]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[3]}} \frac{\partial \mathbf{a}^{[3]}}{\partial \mathbf{z}^{[2]}} = (W^{[3]})^T \delta^{[3]}$$

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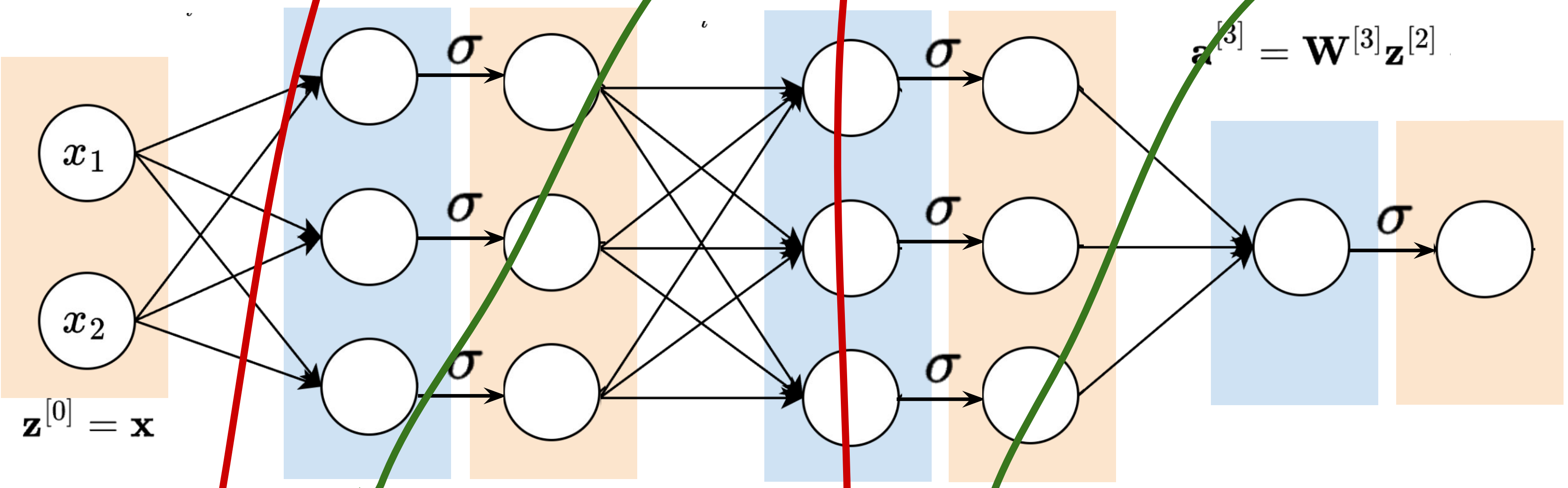
$$\text{Loss} = \mathcal{L}(\mathbf{z}^{[3]}, \mathbf{y})$$

$$\delta^{[1]} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[1]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[1]}} \frac{\partial \mathbf{z}^{[1]}}{\partial \mathbf{a}^{[1]}}$$

$$\delta^{[2]} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[2]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[2]}} \frac{\partial \mathbf{z}^{[2]}}{\partial \mathbf{a}^{[2]}}$$

$$\delta^{[3]} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[3]}} \quad \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}}$$

$$\mathbf{a}^{[1]} = \mathbf{W}^{[1]}\mathbf{z}^{[0]} \qquad \mathbf{a}^{[2]} = \mathbf{W}^{[2]}\mathbf{z}^{[1]} \qquad \mathbf{a}^{[3]} = \mathbf{W}^{[3]}\mathbf{z}^{[2]}$$



$$\mathbf{z}^{[0]} = \mathbf{x} \qquad \mathbf{z}^{[1]} = \sigma(\mathbf{a}^{[1]}) \qquad \mathbf{z}^{[2]} = \sigma(\mathbf{a}^{[2]}) \qquad \mathbf{z}^{[3]} = \sigma(\mathbf{a}^{[3]})$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[1]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[2]}} \frac{\partial \mathbf{a}^{[2]}}{\partial \mathbf{z}^{[1]}} = (W^{[2]})^T \delta^{[2]}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[2]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[3]}} \frac{\partial \mathbf{a}^{[3]}}{\partial \mathbf{z}^{[2]}} = (W^{[3]})^T \delta^{[3]}$$

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Backpropagation

Algorithm Backward Pass through MLP (Detailed)

Input: $\{\mathbf{z}^{[1]}, \ldots, \mathbf{z}^{[L]}\}$, $\{\mathbf{a}^{[1]}, \ldots, \mathbf{a}^{[L]}\}$, loss gradient $\frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[L]}}$

$\delta^{[L]} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[L]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[L]}} \frac{\partial \mathbf{z}^{[L]}}{\partial \mathbf{a}^{[L]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[L]}} \odot \sigma^{[L]\prime}(\mathbf{a}^{[L]})$ ▷ Error term

for $l = L$ **to** 1 **do**

$\quad \frac{\partial \mathcal{L}}{\partial \mathbf{W}^{[l]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[l]}} \frac{\partial \mathbf{a}^{[l]}}{\partial \mathbf{W}^{[l]}} = \delta^{[l]} (\mathbf{z}^{[l-1]})^T$ ▷ Gradient of weights

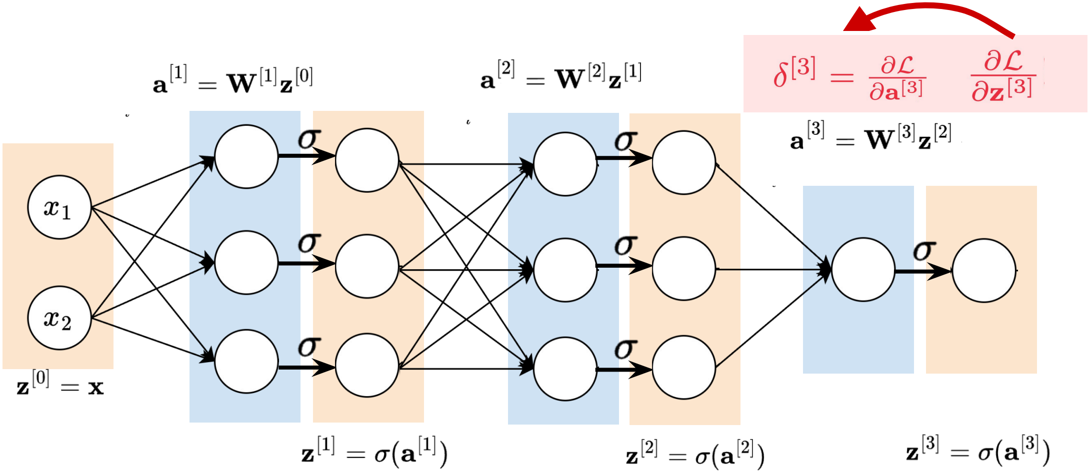
$\quad \frac{\partial \mathcal{L}}{\partial \mathbf{b}^{[l]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[l]}} \frac{\partial \mathbf{a}^{[l]}}{\partial \mathbf{b}^{[l]}} = \delta^{[l]}$ ▷ Gradient of biases

$\quad \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[l-1]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[l]}} \frac{\partial \mathbf{a}^{[l]}}{\partial \mathbf{z}^{[l-1]}} = (\mathbf{W}^{[l]})^T \delta^{[l]}$

$\quad \delta^{[l-1]} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[l-1]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[l-1]}} \frac{\partial \mathbf{z}^{[l-1]}}{\partial \mathbf{a}^{[l-1]}} = ((\mathbf{W}^{[l]})^T \delta^{[l]}) \odot \sigma^{[l-1]\prime}(\mathbf{a}^{[l-1]})$

end for

Output: $\frac{\partial \mathcal{L}}{\partial \mathbf{W}^{[1:L]}}, \frac{\partial \mathcal{L}}{\partial \mathbf{b}^{[1:L]}}$



$\mathcal{L}(\mathbf{z}^{[3]}, \mathbf{y})$

We can directly compute $\frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}}$!

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Backpropagation

$$\frac{\partial \mathcal{L}}{\partial \mathbf{W}^{[3]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[3]}} \frac{\partial \mathbf{a}^{[3]}}{\partial \mathbf{W}^{[3]}} = \delta^{[3]} (\mathbf{z}^{[2]})^T$$

Algorithm Backward Pass through MLP (Detailed)

Input: $\{\mathbf{z}^{[1]}, \ldots, \mathbf{z}^{[L]}\}$, $\{\mathbf{a}^{[1]}, \ldots, \mathbf{a}^{[L]}\}$, loss gradient $\frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[L]}}$

$\delta^{[L]} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[L]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[L]}} \frac{\partial \mathbf{z}^{[L]}}{\partial \mathbf{a}^{[L]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[L]}} \odot \sigma^{[L]\prime}(\mathbf{a}^{[L]})$ ▷ Error term

for $l = L$ **to** 1 **do**

  $\frac{\partial \mathcal{L}}{\partial \mathbf{W}^{[l]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[l]}} \frac{\partial \mathbf{a}^{[l]}}{\partial \mathbf{W}^{[l]}} = \delta^{[l]} (\mathbf{z}^{[l-1]})^T$ ▷ Gradient of weights

  $\frac{\partial \mathcal{L}}{\partial \mathbf{b}^{[l]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[l]}} \frac{\partial \mathbf{a}^{[l]}}{\partial \mathbf{b}^{[l]}} = \delta^{[l]}$ ▷ Gradient of biases

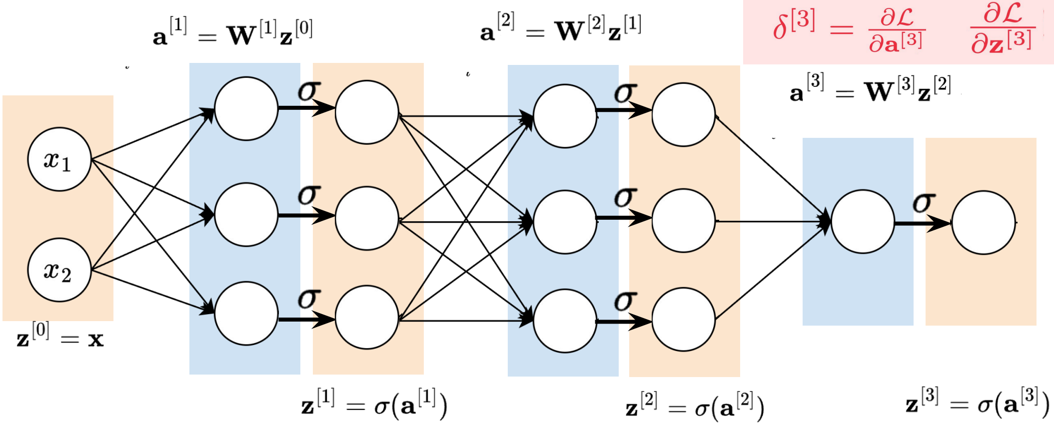
  $\frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[l-1]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[l]}} \frac{\partial \mathbf{a}^{[l]}}{\partial \mathbf{z}^{[l-1]}} = (\mathbf{W}^{[l]})^T \delta^{[l]}$

  $\delta^{[l-1]} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[l-1]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[l-1]}} \frac{\partial \mathbf{z}^{[l-1]}}{\partial \mathbf{a}^{[l-1]}} = ((\mathbf{W}^{[l]})^T \delta^{[l]}) \odot \sigma^{[l-1]\prime}(\mathbf{a}^{[l-1]})$

end for

Output: $\frac{\partial \mathcal{L}}{\partial \mathbf{W}^{[1:L]}}, \frac{\partial \mathcal{L}}{\partial \mathbf{b}^{[1:L]}}$

$\delta^{[3]} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[3]}} \quad \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}}$



$$\mathcal{L}(\mathbf{z}^{[3]}, \mathbf{y})$$

We can directly compute $\frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}}$!

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Backpropagation

$$\frac{\partial \mathcal{L}}{\partial \mathbf{W}^{[3]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[3]}} \frac{\partial \mathbf{a}^{[3]}}{\partial \mathbf{W}^{[3]}} = \delta^{[3]} (\mathbf{z}^{[2]})^T$$

Algorithm Backward Pass through MLP (Detailed)

Input: $\{\mathbf{z}^{[1]}, \ldots, \mathbf{z}^{[L]}\}$, $\{\mathbf{a}^{[1]}, \ldots, \mathbf{a}^{[L]}\}$, loss gradient $\frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[L]}}$

$\delta^{[L]} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[L]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[L]}} \frac{\partial \mathbf{z}^{[L]}}{\partial \mathbf{a}^{[L]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[L]}} \odot \sigma^{[L]\prime}(\mathbf{a}^{[L]})$ ▷ Error term

for $l = L$ **to** 1 **do**

  $\frac{\partial \mathcal{L}}{\partial \mathbf{W}^{[l]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[l]}} \frac{\partial \mathbf{a}^{[l]}}{\partial \mathbf{W}^{[l]}} = \delta^{[l]} (\mathbf{z}^{[l-1]})^T$ ▷ Gradient of weights

  $\frac{\partial \mathcal{L}}{\partial \mathbf{b}^{[l]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[l]}} \frac{\partial \mathbf{a}^{[l]}}{\partial \mathbf{b}^{[l]}} = \delta^{[l]}$ ▷ Gradient of biases

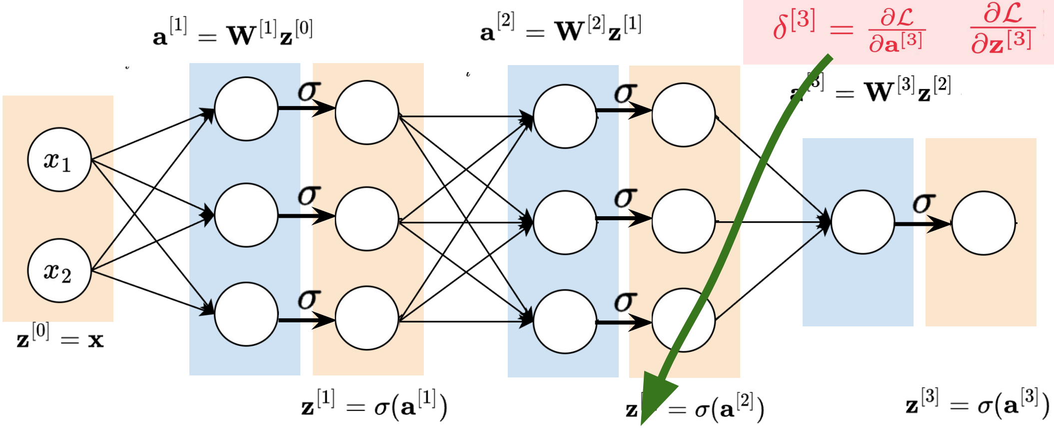
  $\frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[l-1]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[l]}} \frac{\partial \mathbf{a}^{[l]}}{\partial \mathbf{z}^{[l-1]}} = (\mathbf{W}^{[l]})^T \delta^{[l]}$

  $\delta^{[l-1]} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[l-1]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[l-1]}} \frac{\partial \mathbf{z}^{[l-1]}}{\partial \mathbf{a}^{[l-1]}} = ((\mathbf{W}^{[l]})^T \delta^{[l]}) \odot \sigma^{[l-1]\prime}(\mathbf{a}^{[l-1]})$

end for

Output: $\frac{\partial \mathcal{L}}{\partial \mathbf{W}^{[1:L]}}, \frac{\partial \mathcal{L}}{\partial \mathbf{b}^{[1:L]}}$

$\delta^{[3]} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[3]}} \quad \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}}$



$$\frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[2]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[3]}} \frac{\partial \mathbf{a}^{[3]}}{\partial \mathbf{z}^{[2]}} = (W^{[3]})^T \delta^{[3]}$$

$$\mathcal{L}(\mathbf{z}^{[3]}, \mathbf{y})$$

We can directly compute $\frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}}$!

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Backpropagation

$$\frac{\partial \mathcal{L}}{\partial \mathbf{W}^{[3]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[3]}} \frac{\partial \mathbf{a}^{[3]}}{\partial \mathbf{W}^{[3]}} = \delta^{[3]} (\mathbf{z}^{[2]})^T$$

Algorithm Backward Pass through MLP (Detailed)

Input: $\{\mathbf{z}^{[1]}, \ldots, \mathbf{z}^{[L]}\}$, $\{\mathbf{a}^{[1]}, \ldots, \mathbf{a}^{[L]}\}$, loss gradient $\frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[L]}}$

$\delta^{[L]} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[L]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[L]}} \frac{\partial \mathbf{z}^{[L]}}{\partial \mathbf{a}^{[L]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[L]}} \odot \sigma^{[L]\prime}(\mathbf{a}^{[L]})$ ▷ Error term

for $l = L$ **to** 1 **do**

$\quad \frac{\partial \mathcal{L}}{\partial \mathbf{W}^{[l]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[l]}} \frac{\partial \mathbf{a}^{[l]}}{\partial \mathbf{W}^{[l]}} = \delta^{[l]} (\mathbf{z}^{[l-1]})^T$ ▷ Gradient of weights

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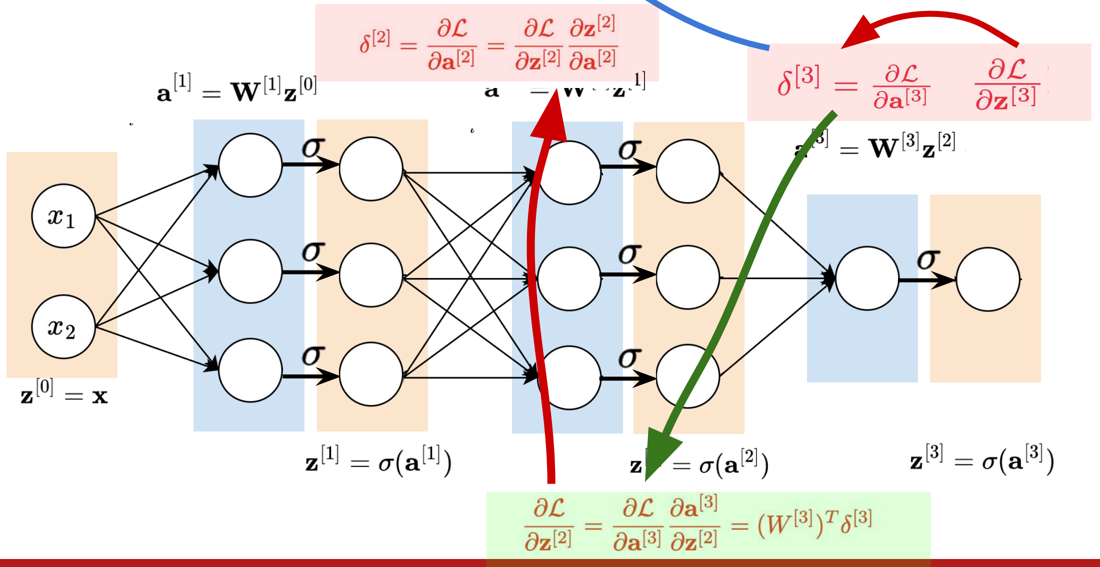
$\quad \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[l-1]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[l]}} \frac{\partial \mathbf{a}^{[l]}}{\partial \mathbf{z}^{[l-1]}} = (\mathbf{W}^{[l]})^T \delta^{[l]}$

$\quad \delta^{[l-1]} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}^{[l-1]}} = \frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[l-1]}} \frac{\partial \mathbf{z}^{[l-1]}}{\partial \mathbf{a}^{[l-1]}} = ((\mathbf{W}^{[l]})^T \delta^{[l]}) \odot \sigma^{[l-1]\prime}(\mathbf{a}^{[l-1]})$

end for

Output: $\frac{\partial \mathcal{L}}{\partial \mathbf{W}^{[1:L]}}, \frac{\partial \mathcal{L}}{\partial \mathbf{b}^{[1:L]}}$



$$\mathcal{L}(\mathbf{z}^{[3]}, \mathbf{y})$$

We can directly compute $\frac{\partial \mathcal{L}}{\partial \mathbf{z}^{[3]}}$!

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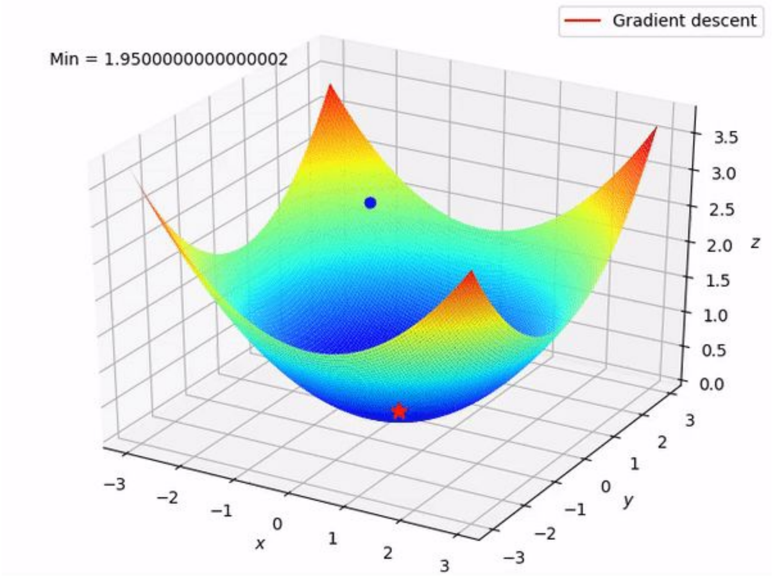
What is Optimization?



In deep learning, optimization methods attempt to find model weights that **minimize the loss function**.

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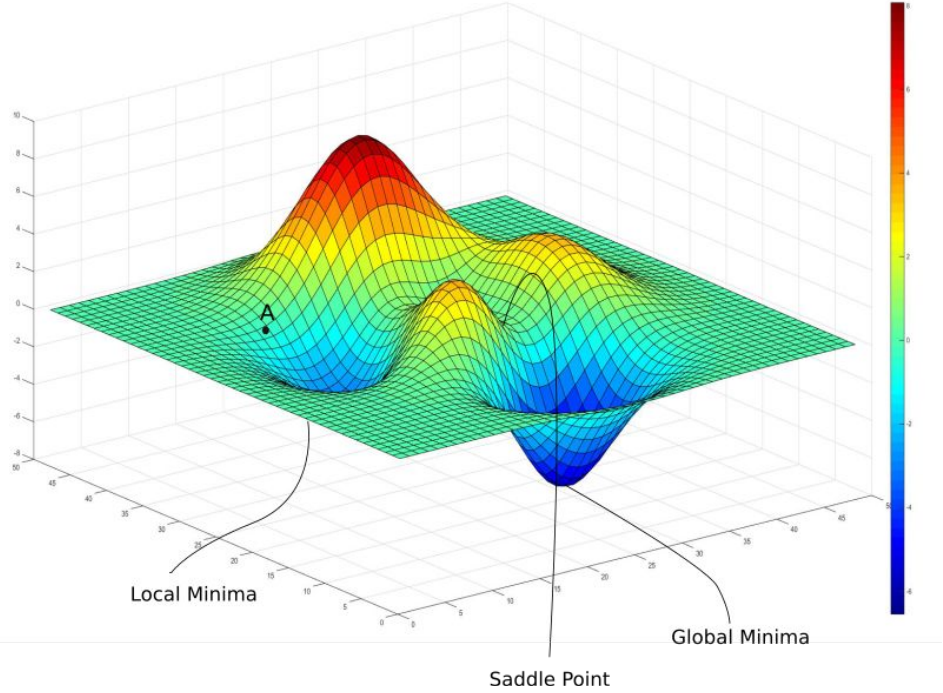
Loss function

Empirical Risk:

$$\mathcal{L}(\mathbf{w}_t) = \frac{1}{n} \sum_{i=1,\ldots,n} \ell(\mathbf{w}_t, \mathbf{x}_i)$$

t : at time step t
$\mathbf{w}_t$: Model weights (parameters) at time t
$\mathbf{x}_i$: The i-th input training data

$\mathcal{L}$: the Loss function (optimization target)
ℓ : per-sample loss



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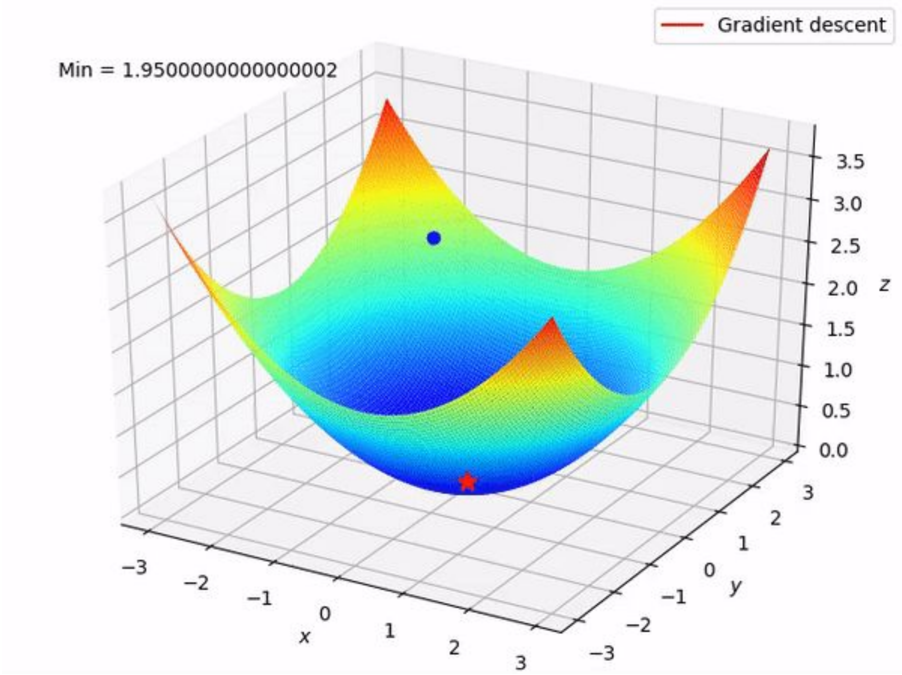
Gradient Descent (GD)

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \nabla \mathcal{L}(\mathbf{w}_t)$$

α: the learning rate

$\nabla \mathcal{L}(\mathbf{w}_t)$: the gradient of Loss w.r.t. $\mathbf{w}_t$



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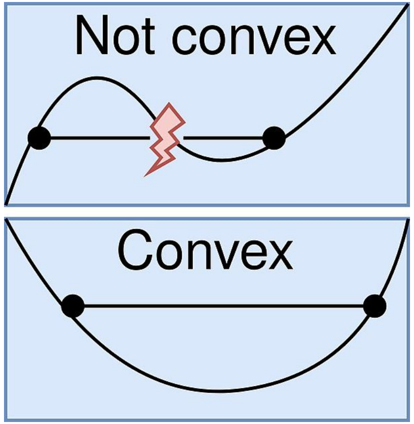
Demo

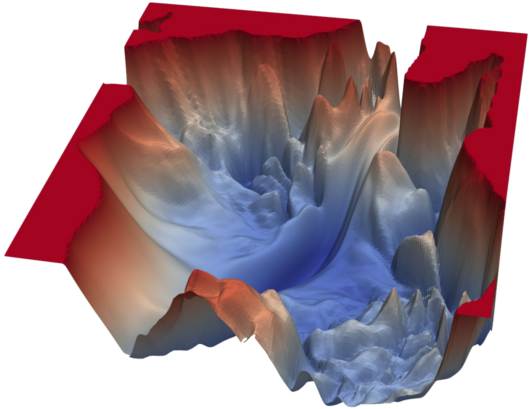
Gradient descent with global minimum

What are some potential problems with gradient descent?

Convexity

- A function on a graph is **convex** if a line segment drawn through any two points on the line of the function, then it never lies below the curved line segment
- Convexity implies that every local minimum is **global minimum**.
- Neural networks are **not** convex!





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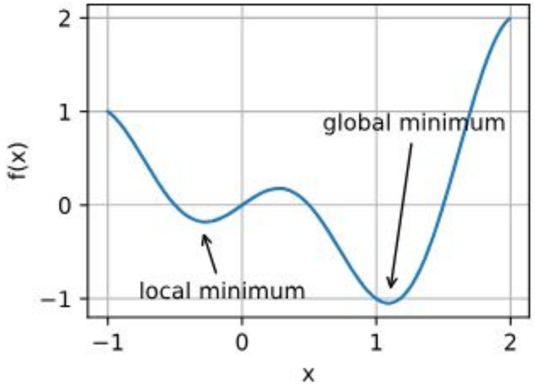
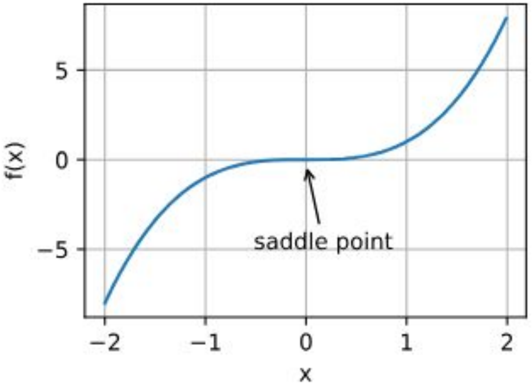
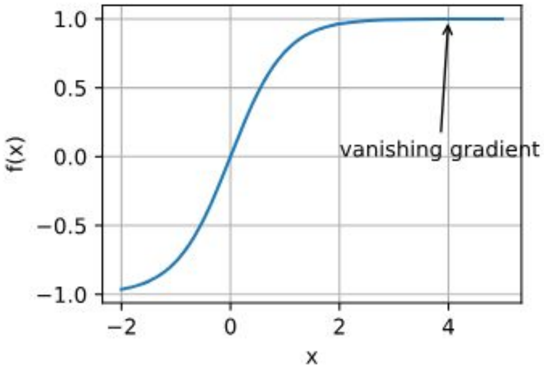
Challenges in Non-Convex Optimization



Local Minima vs. Global Minima



Saddle Points



Vanishing gradient

Demo

Gradient descent with local minimum

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Gradient Descent (GD)

$$\mathcal{L}(\mathbf{w}_t) = \frac{1}{n}\sum_{i=1}^{n} \ell(\mathbf{w}_t, \mathbf{x}_i)$$

$$\nabla\mathcal{L}(\mathbf{w}_t) = \frac{1}{n}\sum_{i=1}^{n} \nabla\ell(\mathbf{w}_t, \mathbf{x}_i)$$

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha\nabla\mathcal{L}(\mathbf{w}_t)$$

Full gradient: $\mathcal{O}(n)$ time => ***Too expensive!***

- *Statistically, why don't we use 1 or a few samples from the training dataset to approximate the full gradient?*

Gradient Descent (GD)

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \nabla \mathcal{L}(\mathbf{w}_t)$$

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Gradient Descent (GD)

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \nabla \mathcal{L}(\mathbf{w}_t)$$

Select **1** example randomly each time

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Gradient Descent (GD)

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \nabla \mathcal{L}(\mathbf{w}_t)$$

Select **1** example randomly each time

Per-sample gradient is equivalent to full gradient in expectation!

$$\mathbb{E}[\nabla \ell(\mathbf{w}_t, \mathbf{x}_i)] = \frac{1}{n} \sum_{i=1}^{n} \nabla \ell(\mathbf{w}_t, \mathbf{x}_i) = \nabla \mathcal{L}(\mathbf{w}_t)$$

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Stochastic Gradient Descent (SGD)

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \nabla \mathcal{L}(\mathbf{w}_t)$$

Select **1** example randomly each time

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \nabla \ell(\mathbf{w}_t, \mathbf{x}_i)$$

Per-sample gradient is equivalent to full gradient in expectation!

$$\mathbb{E}[\nabla \ell(\mathbf{w}_t, \mathbf{x}_i)] = \frac{1}{n} \sum_{i=1}^{n} \nabla \ell(\mathbf{w}_t, \mathbf{x}_i) = \nabla \mathcal{L}(\mathbf{w}_t)$$

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Stochastic Gradient Descent (SGD)

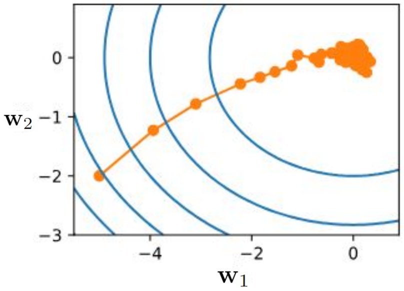
$$\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \nabla \mathcal{L}(\mathbf{w}_t)$$

Select **1** example randomly each time

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \nabla \ell(\mathbf{w}_t, \mathbf{x}_i)$$

Trade off convergence*!*

Per-sample gradients not necessarily points to the local minimum, introducing a ***noise ball****…*



Stochastic Gradient Descent (SGD)

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \nabla \mathcal{L}(\mathbf{w}_t)$$

↓ Select **1** example randomly each time

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \nabla \ell(\mathbf{w}_t, \mathbf{x}_i)$$

↓ Select a batch $\mathcal{B}_t$ of examples randomly each time, with *batch size* b

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Minibatch SGD

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \nabla \mathcal{L}(\mathbf{w}_t)$$

Select **1** example randomly each time

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \nabla \ell(\mathbf{w}_t, \mathbf{x}_i)$$

Select a batch $\mathcal{B}_t$ of examples randomly each time, with *batch size* b

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \cdot \frac{1}{b} \sum_{i \in \mathcal{B}_t} \nabla \ell(\mathbf{w}_t, \mathbf{x}_i)$$

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Let's look at an example!

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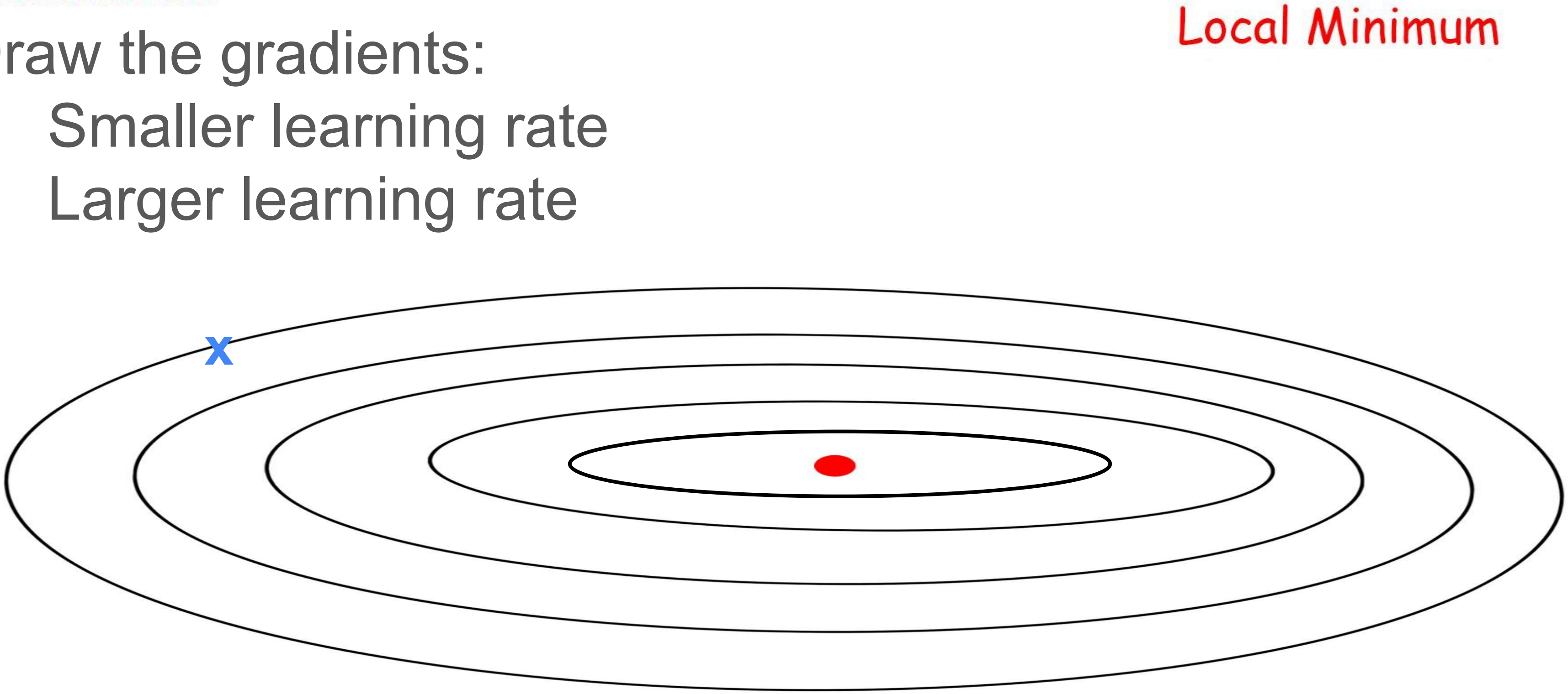
Draw the gradients:

- Smaller learning rate
- Larger learning rate



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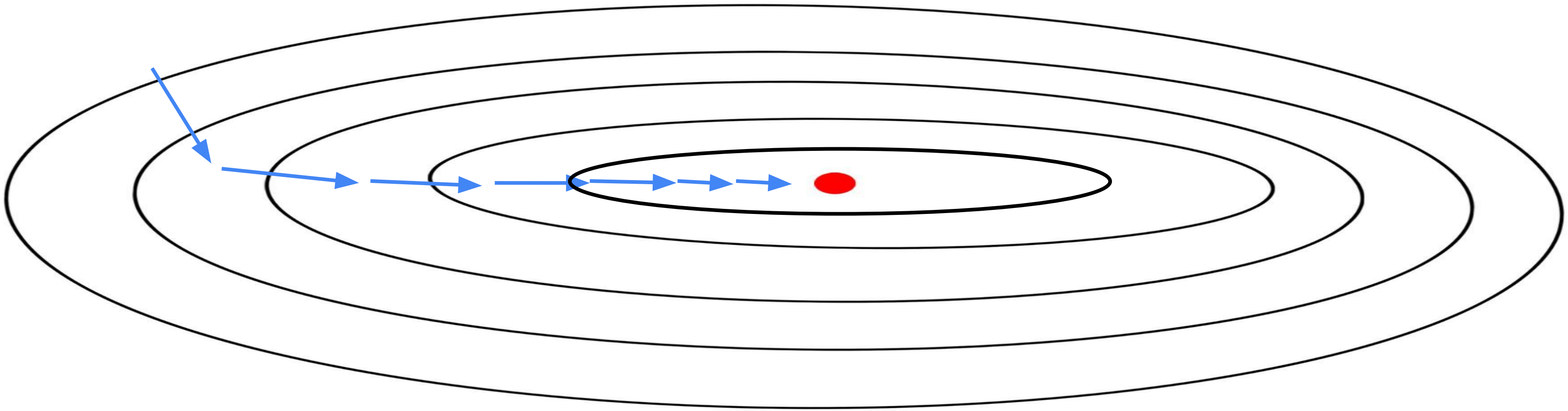
Local Minimum



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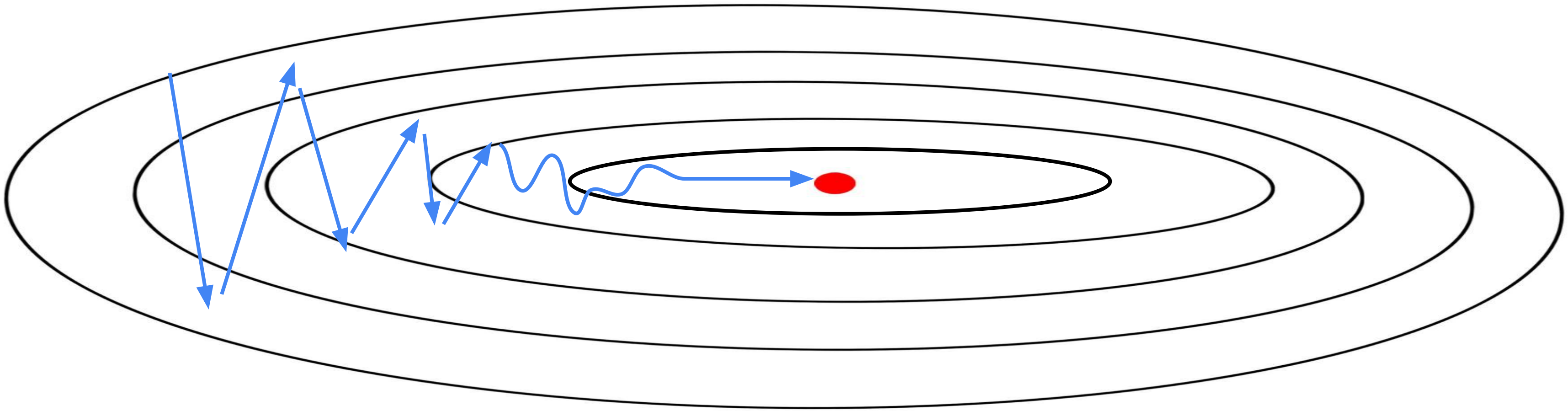
Local Minimum



Cornell Bowers C·IS

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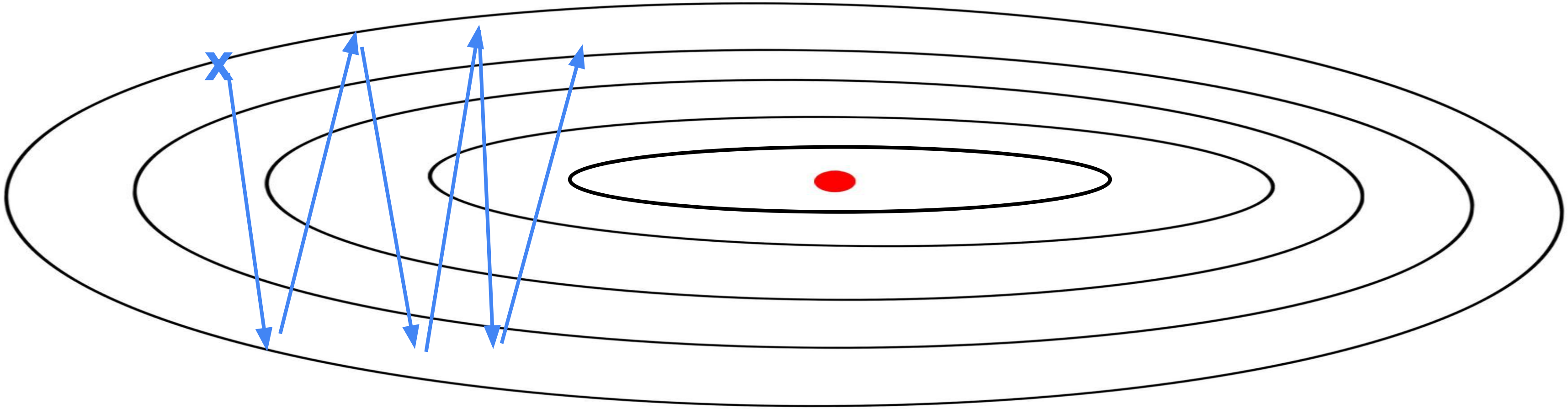
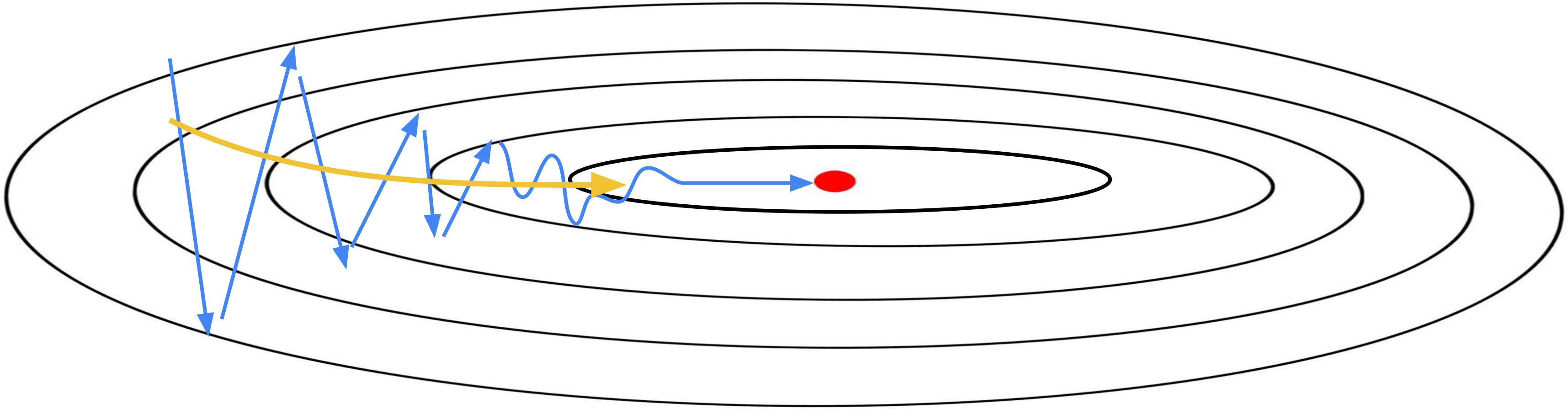


Cornell Bowers C·IS

Local Minimum
Minibatch SGD
Momentum



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SGD with Momentum (Polyak, 1964)

*Compute an **Exponentially Weighted Moving Average (EWMA)** of the gradients as **momentum** and use that to update the weight instead.*

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SGD Update Rule

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \nabla \ell(\mathbf{w}_t, \mathbf{x}_i)$$

→

$$\mathbf{m}_{t+1} = \mu \mathbf{m}_t - \alpha \nabla l(\mathbf{w}_t; \mathbf{x}_i)$$
$$\mathbf{w}_{t+1} = \mathbf{w}_t + \mathbf{m}_{t+1}$$

where $\mu \in [0, 1]$ is the momentum coefficient.

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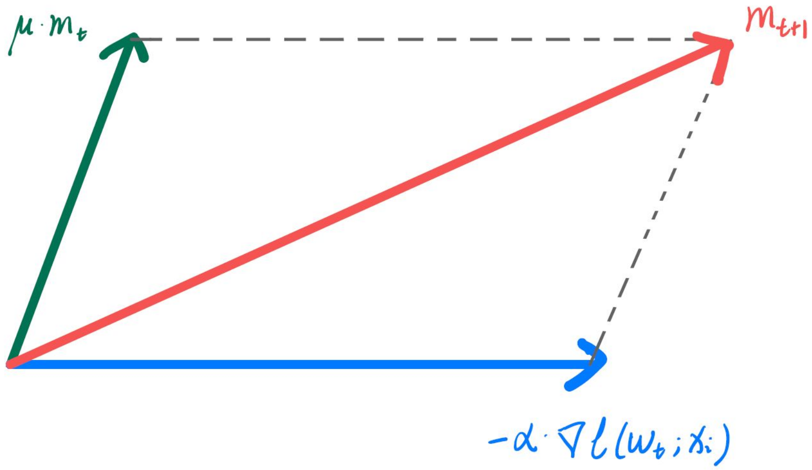

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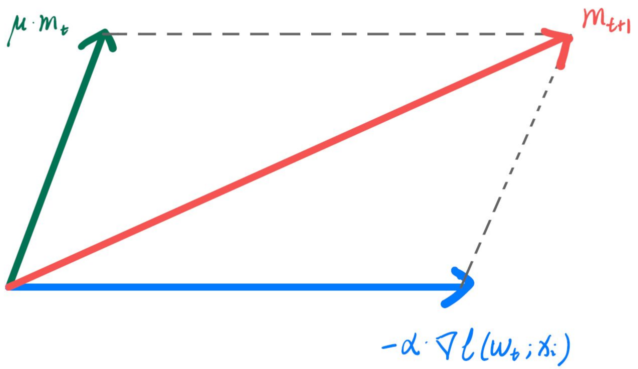

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SGD with Momentum

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$$\mathbf{g}_t = \nabla l(\mathbf{w}_t; \mathbf{x}_i)$$
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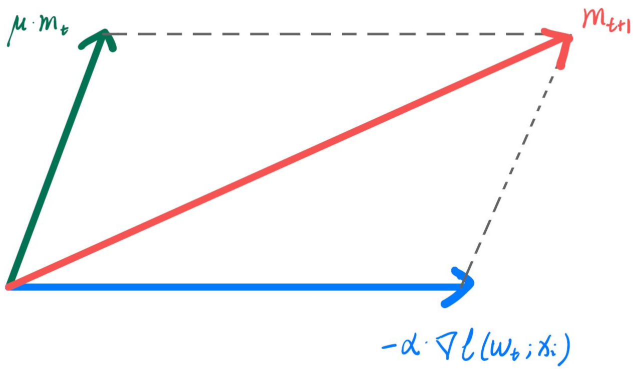

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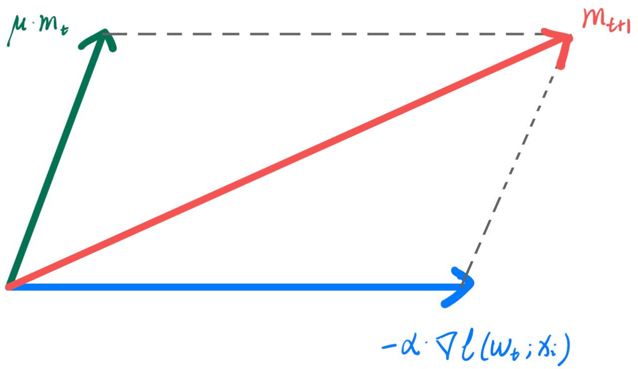

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$$
\begin{aligned}
\mathbf{g}_t &= \nabla l(\mathbf{w}_t; \mathbf{x}_i) \\
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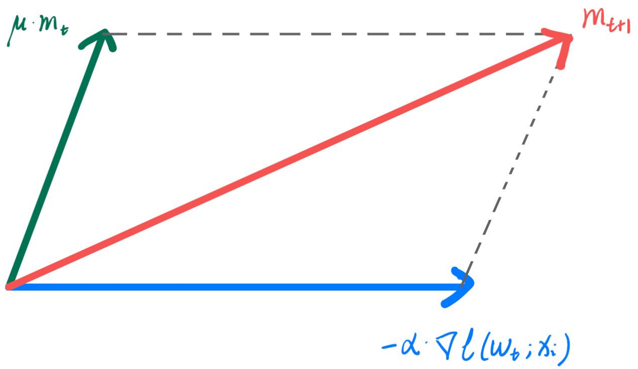
\mathbf{w}_{t+1} &= \mathbf{w}_t + \mu \mathbf{m}_t - \alpha \mathbf{g}_t \\
&= \mathbf{w}_t + \mu(\mu \mathbf{m}_{t-1} - \alpha \mathbf{g}_{t-1}) - \alpha \mathbf{g}_t
\end{aligned}
$$



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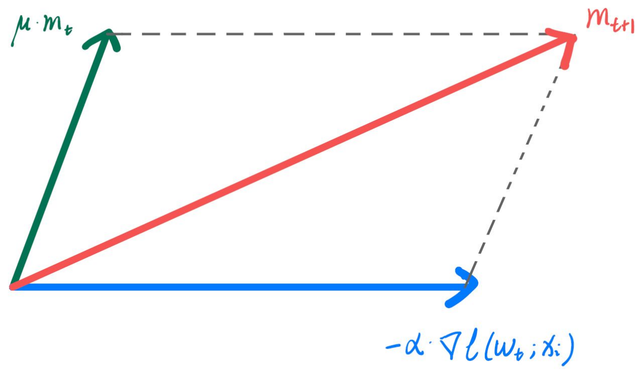
&= \mathbf{w}_t + \mu(\mu \mathbf{m}_{t-1} - \alpha \mathbf{g}_{t-1}) - \alpha \mathbf{g}_t \\
&= \mathbf{w}_t + \mu(\mu(\mu \mathbf{m}_{t-2} - \alpha \mathbf{g}_{t-2}) - \alpha \mathbf{g}_{t-1}) - \alpha \mathbf{g}_t \\
&= \mathbf{w}_t - \alpha \mathbf{g}_t - \mu \alpha \mathbf{g}_{t-1} - \mu^2 \alpha \mathbf{g}_{t-2} - \mu^3 \alpha \mathbf{g}_{t-3} - \ldots
\end{aligned}$$



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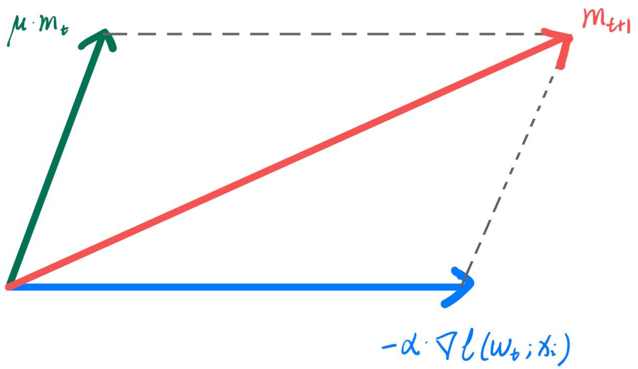

$$
\begin{aligned}
\mathbf{g}_t &= \nabla l(\mathbf{w}_t; \mathbf{x}_i) \\
\mathbf{m}_{t+1} &= \mu \mathbf{m}_t - \alpha \mathbf{g}_t \\
\mathbf{w}_{t+1} &= \mathbf{w}_t + \mu \mathbf{m}_t - \alpha \mathbf{g}_t \\
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&= \mathbf{w}_t + \mu(\mu(\mu \mathbf{m}_{t-2} - \alpha \mathbf{g}_{t-2}) - \alpha \mathbf{g}_{t-1}) - \alpha \mathbf{g}_t \\
&= \mathbf{w}_t - \alpha \mathbf{g}_t - \mu \alpha \mathbf{g}_{t-1} - \mu^2 \alpha \mathbf{g}_{t-2} - \mu^3 \alpha \mathbf{g}_{t-3} - \dots \\
&= \mathbf{w}_t - \alpha \sum_{i=0}^{t} \mu^i \mathbf{g}_{t-i}
\end{aligned}
$$

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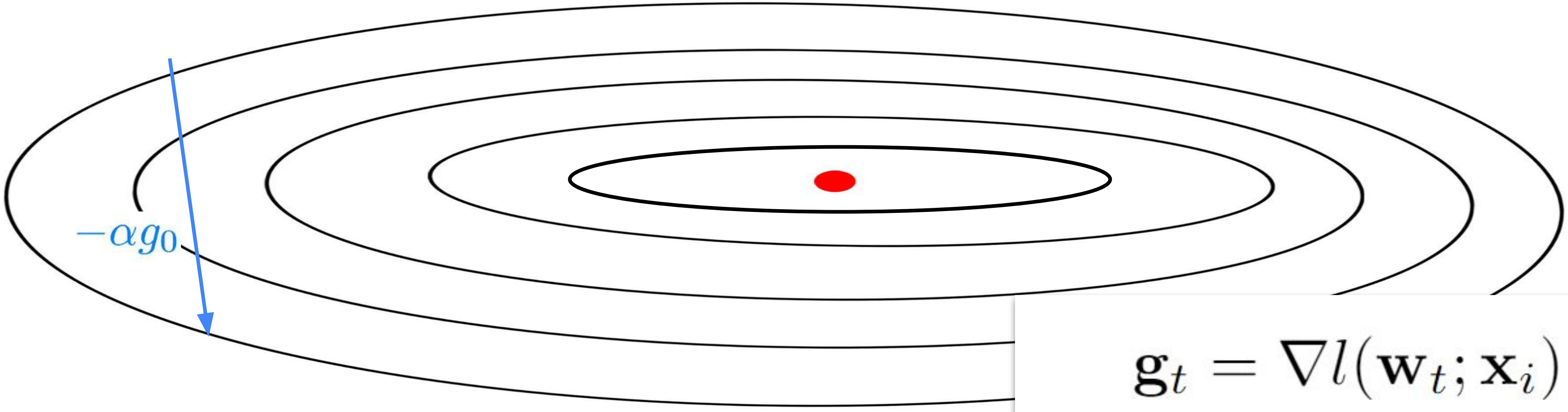
Local Minimum

Minibatch SGD



$$\mathbf{g}_t = \nabla l(\mathbf{w}_t; \mathbf{x}_i)$$

$$\mathbf{m}_{t+1} = \mu \mathbf{m}_t - \alpha \mathbf{g}_t$$

$$\mathbf{w}_{t+1} = \mathbf{w}_t + \mathbf{m}_{t+1}$$

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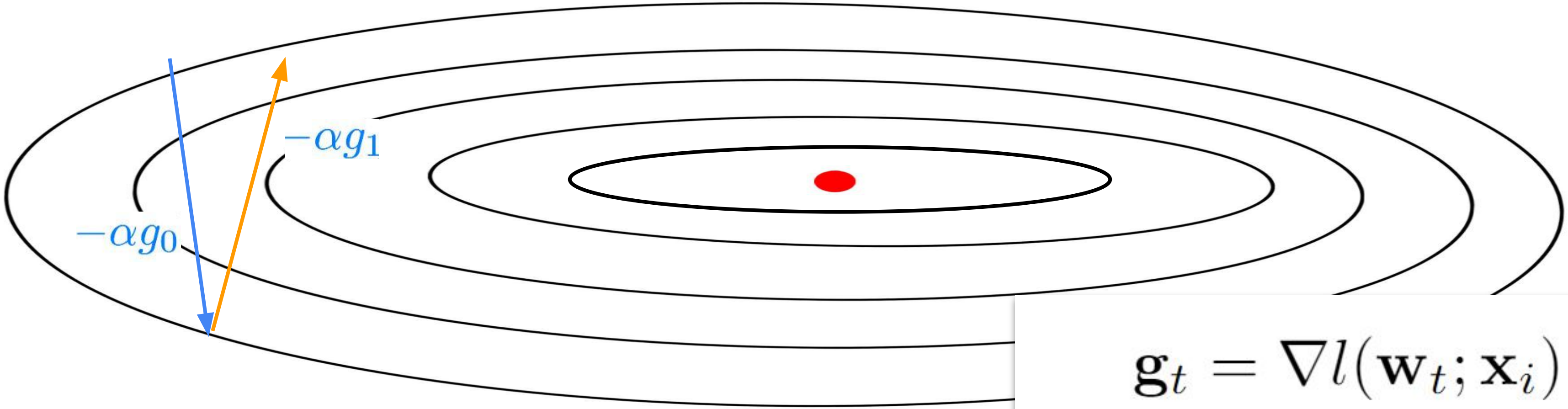
Local Minimum

Minibatch SGD

$$\mathbf{m}_2 = \mu \mathbf{m}_1 - \alpha \mathbf{g}_1$$



$$\mathbf{g}_t = \nabla l(\mathbf{w}_t; \mathbf{x}_i)$$

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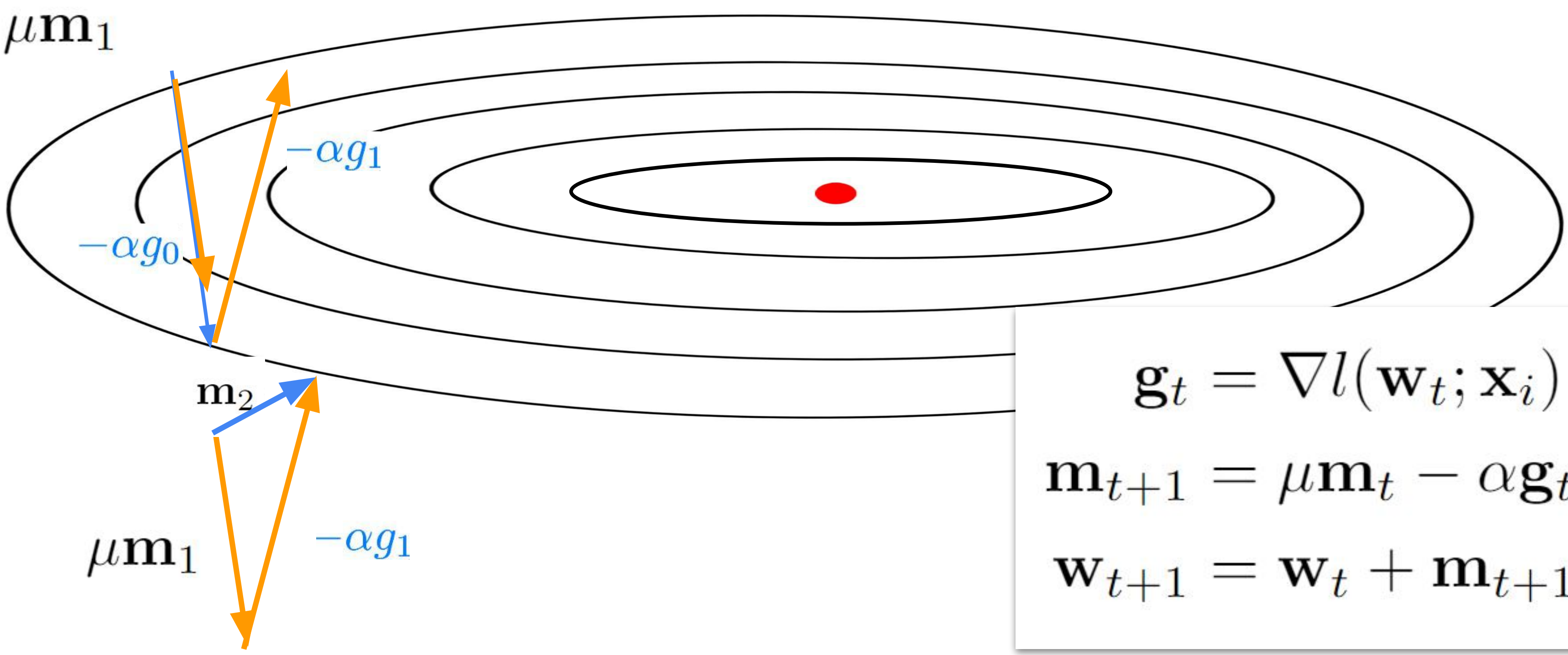
Local Minimum
Minibatch SGD

$$\mathbf{m}_2 = \mu\mathbf{m}_1 - \alpha\mathbf{g}_1$$



$$\mathbf{g}_t = \nabla l(\mathbf{w}_t; \mathbf{x}_i)$$

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Minibatch SGD

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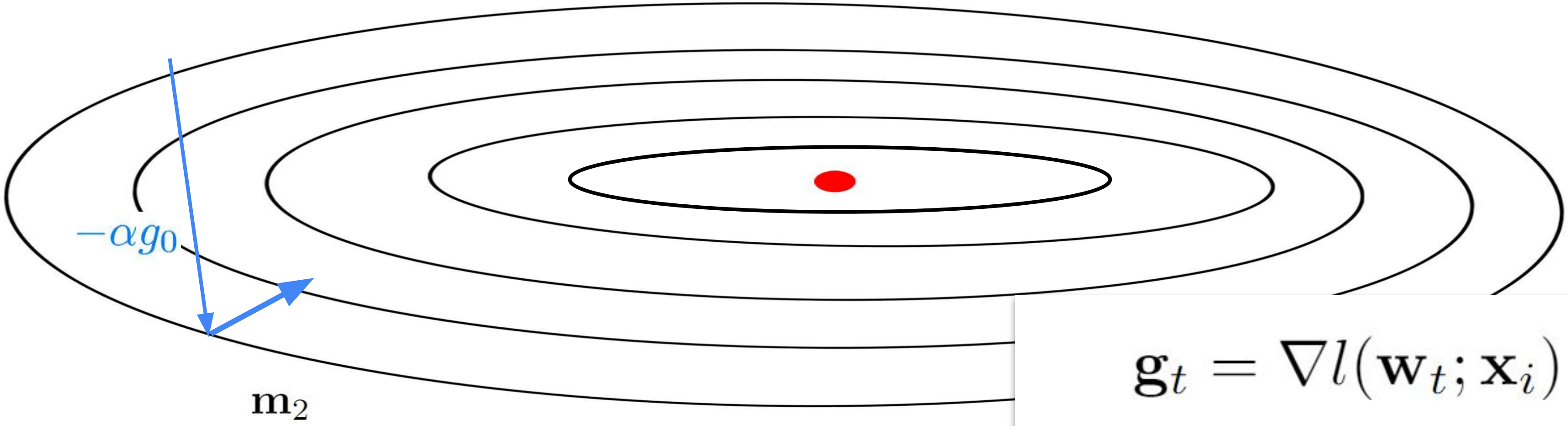

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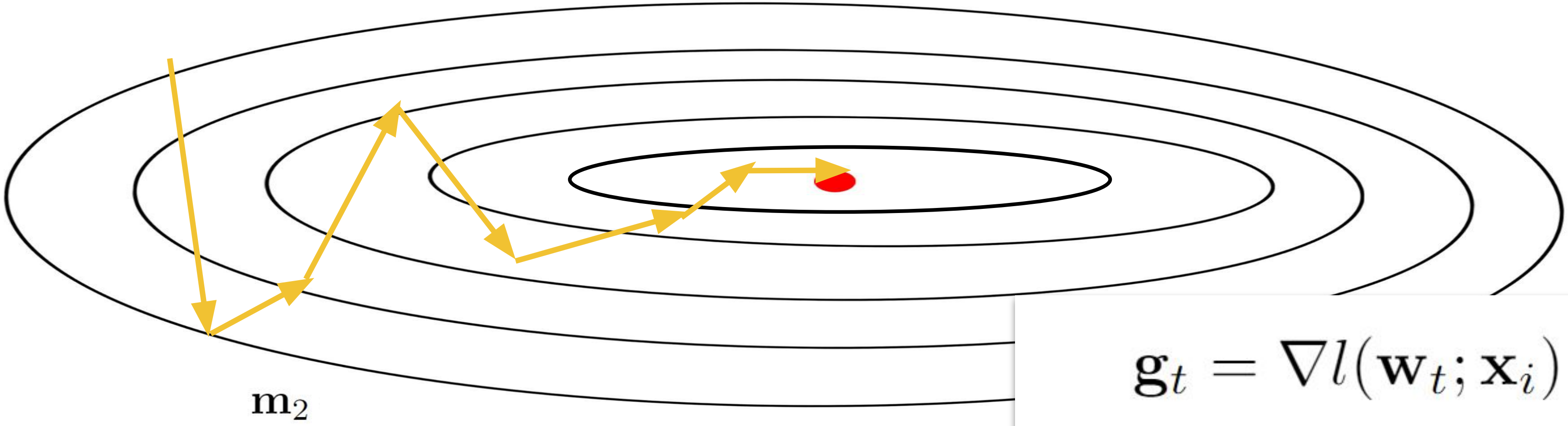
Local Minimum
Minibatch SGD
Momentum



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Local Minimum
Minibatch SGD
Momentum

$$\mathbf{m}_2 = \mu \mathbf{m}_1 - \alpha \mathbf{g}_1$$

Momentum converges almost always faster than standard SGD!

$$\mathbf{g}_t = \nabla l(\mathbf{w}_t; \mathbf{x}_i)$$

$$\mathbf{m}_{t+1} = \mu \mathbf{m}_t - \alpha \mathbf{g}_t$$

$$\mathbf{w}_{t+1} = \mathbf{w}_t + \mathbf{m}_{t+1}$$

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Quick Recap

Gradient Descent

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \cdot \frac{1}{n} \sum_{i=1}^{n} \nabla \ell(\mathbf{w}_t, \mathbf{x}_i)$$

Stochastic Gradient Descent

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \nabla \ell(\mathbf{w}_t, \mathbf{x}_i)$$

Minibatch SGD

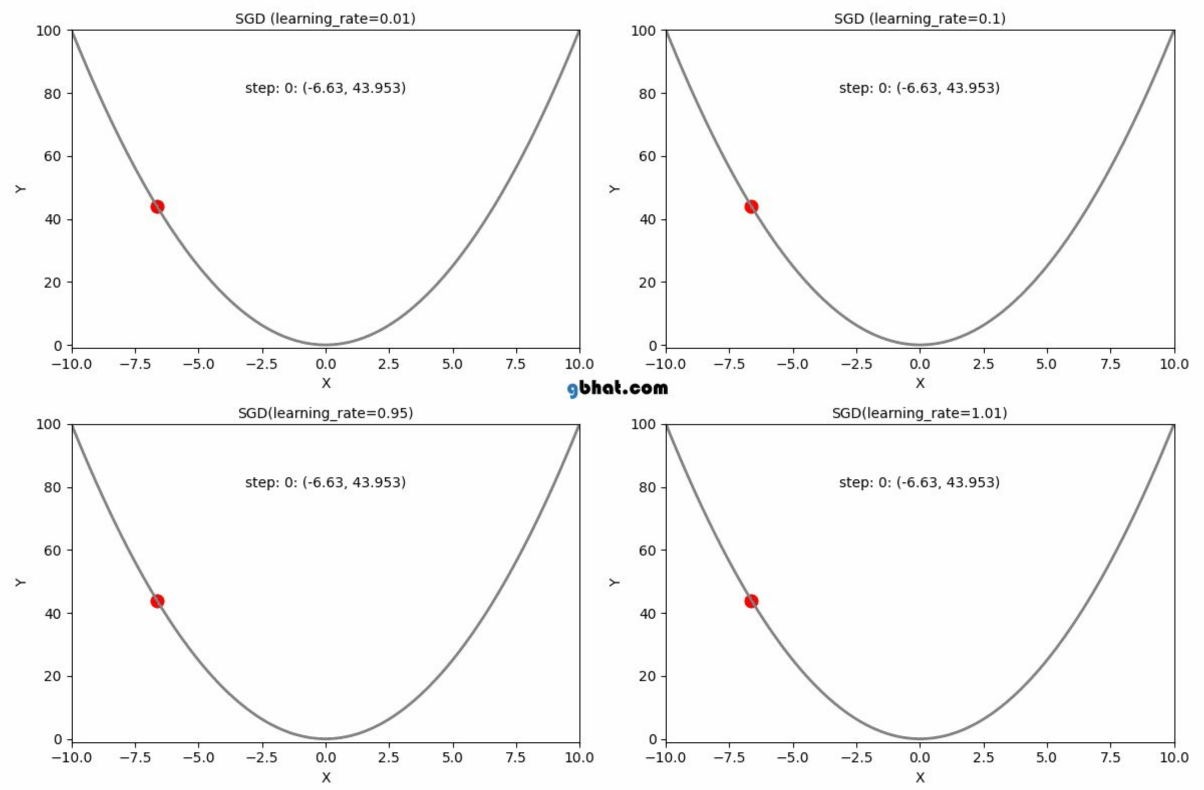
$$\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \cdot \frac{1}{b} \sum_{i \in \mathcal{B}_t} \nabla \ell(\mathbf{w}_t, \mathbf{x}_i)$$

SGD w. Momentum

$$m_{t+1} = \mu m_t - \alpha \nabla l(w_t; x_i)$$

$$w_{t+1} = w_t + m_{t+1}$$

Importance of Learning Rate



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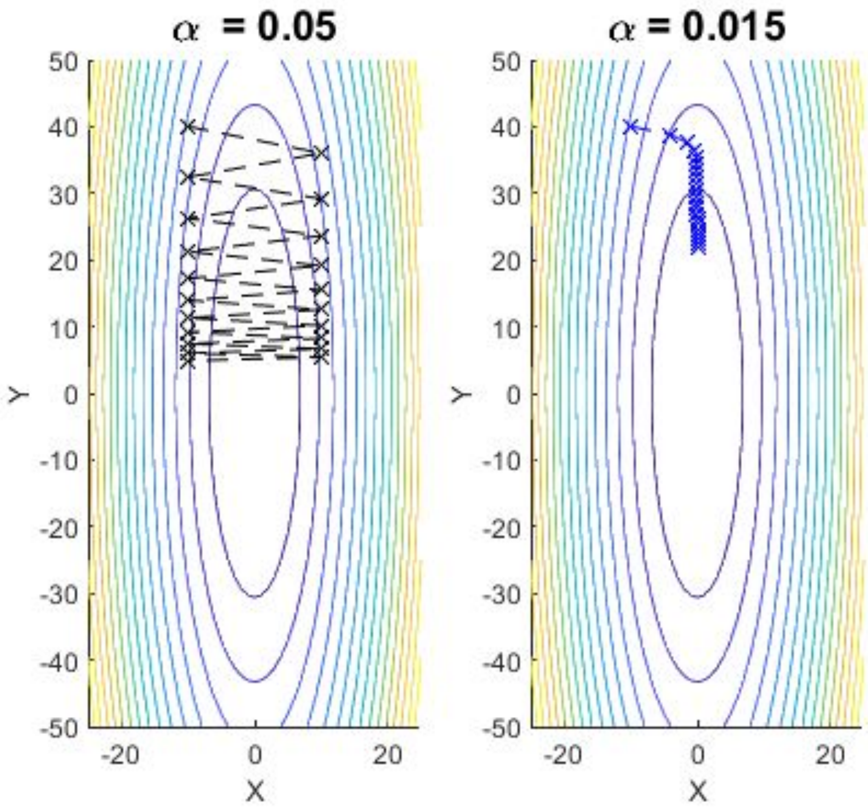
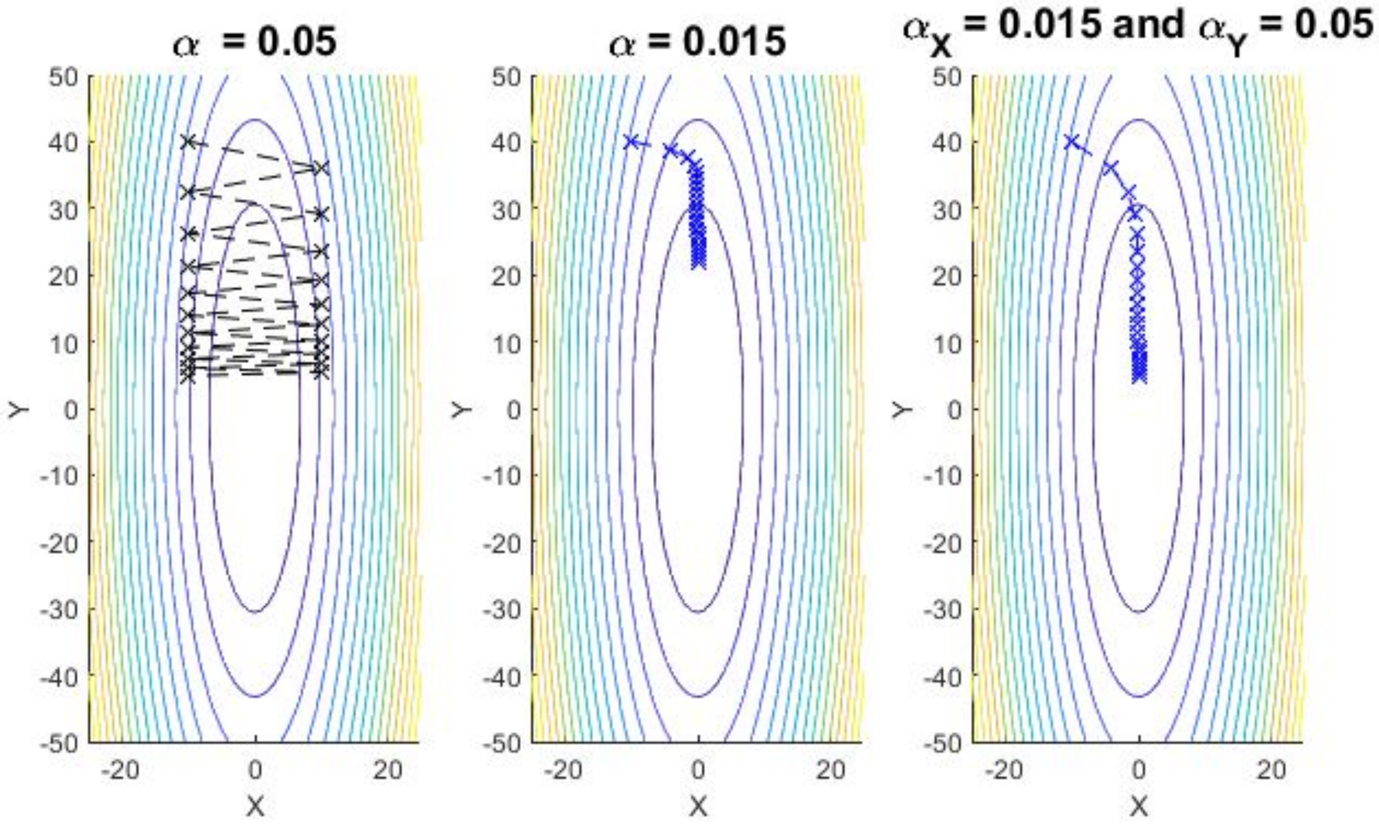
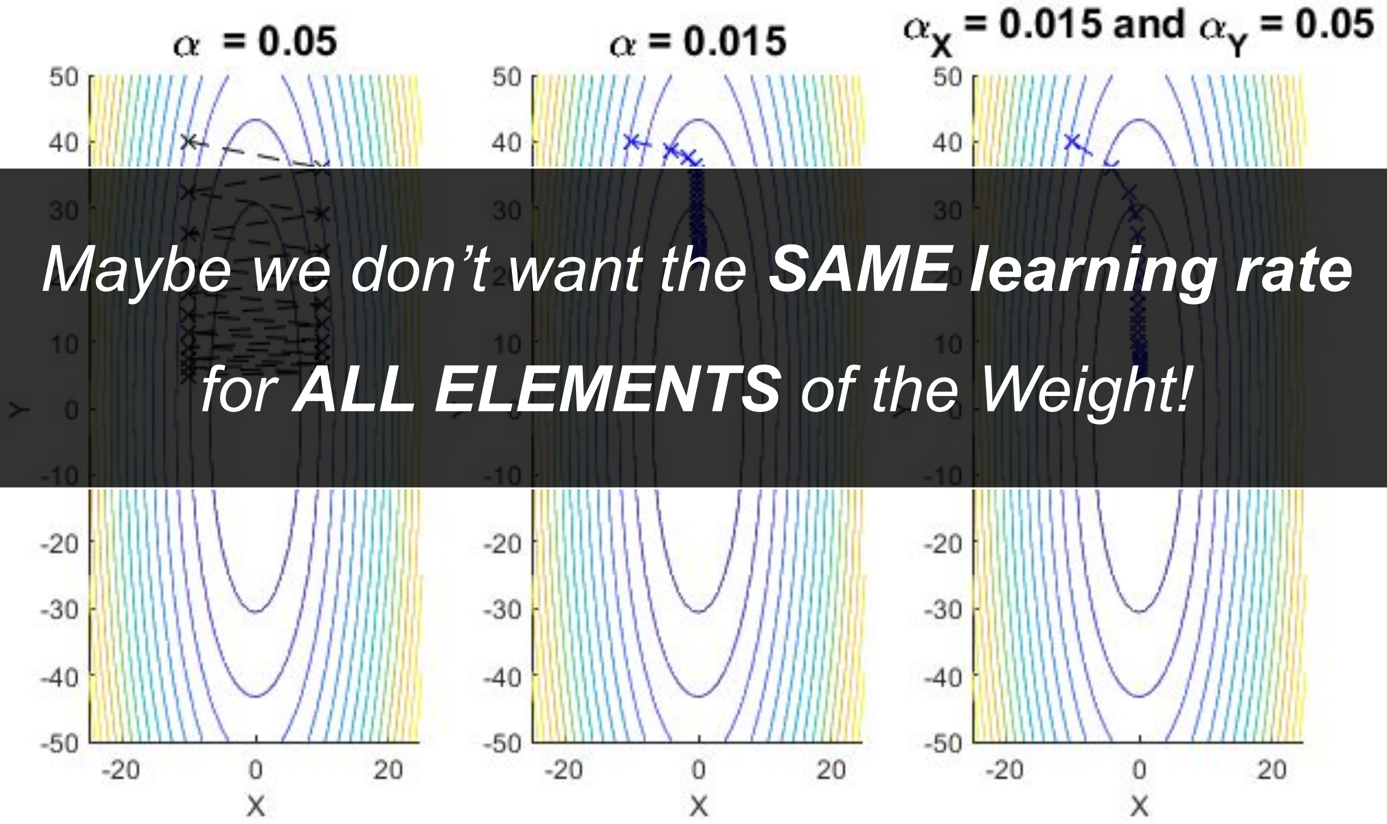
Another example



Another example



Adaptive Learning Rate



Maybe we don't want the ***SAME learning rate***

for ***ALL ELEMENTS*** *of the Weight!*

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Adaptive Optimizers

Different Learning Rate *for* ***each element*** *of the* ***Model Weights!***

AdaGrad (Duchi et al. 2011)

More updates -> more decay

- Handle sparse gradients well
  - *Sparse: The vector has 0 in most of the entries*

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \nabla \mathcal{L}(\mathbf{w}_t)$$

SGD

Element-wise product

$$\mathbf{v}_{t+1} = \mathbf{v}_t + \mathbf{g}_t^2$$

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \frac{\alpha}{\sqrt{\mathbf{v}_{t+1} + \epsilon}} \odot \mathbf{g}_t$$

Adagrad

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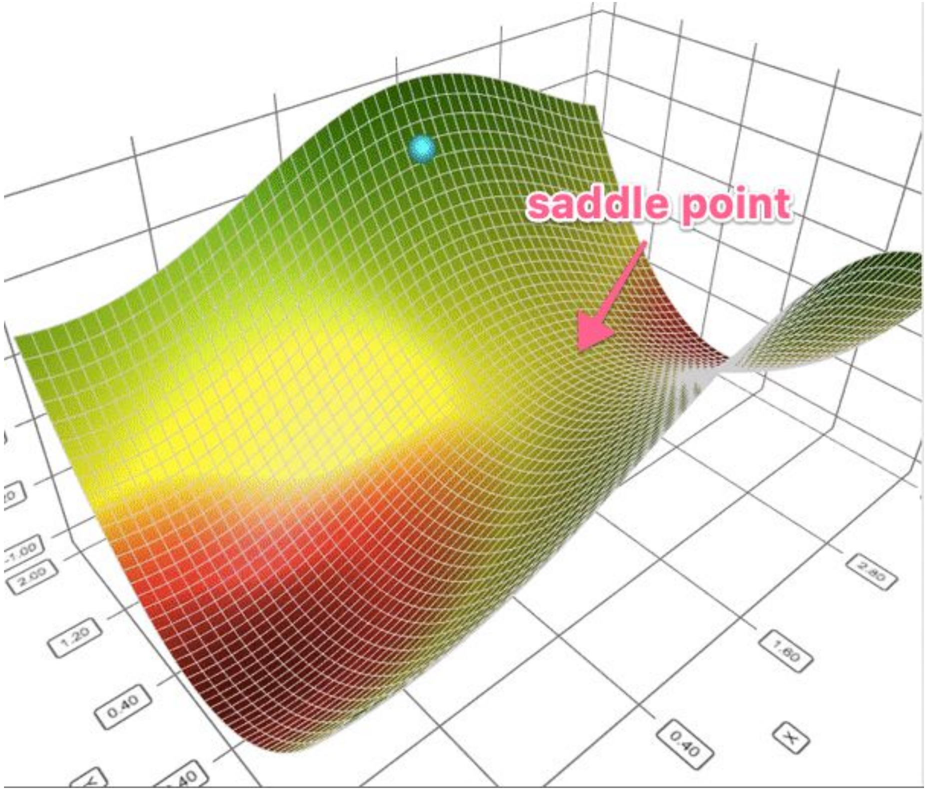

Gradient Descent
AdaGrad

AdaGrad (Duchi et al. 2011)

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Adagrad

Exercise:
What's could be wrong with this optimizator?
(What would happen to the denominator.)

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AdaGrad (Duchi et al. 2011)

More updates → more decay

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$$\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \nabla \mathcal{L}(\mathbf{w}_t)$$

SGD

Element-wise product

$$\mathbf{v}_{t+1} = \mathbf{v}_t + \mathbf{g}_t^2$$

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \frac{\alpha}{\sqrt{\mathbf{v}_{t+1} + \epsilon}} \odot \mathbf{g}_t$$

Adagrad

***Issue**: decays too aggressively!*

RMSProp (Graves, 2013)

Keep an **exponential moving average** of the squared gradient for each element

$$\mathbf{v}_{t+1} = \mathbf{v}_t + \mathbf{g}_t^2$$
$$\mathbf{w}_{t+1} = \mathbf{w}_t - \frac{\alpha}{\sqrt{\mathbf{v}_{t+1} + \epsilon}} \odot \mathbf{g}_t$$

Adagrad

$$\mathbf{v}_{t+1} = \beta \mathbf{v}_t + (1-\beta)\mathbf{g}_t^2$$
$$\mathbf{w}_{t+1} = \mathbf{w}_t - \frac{\alpha}{\sqrt{\mathbf{v}_{t+1} + \epsilon}} \odot \mathbf{g}_t$$

RmsProp

where $\beta \in [0,1]$ the exponential moving average constant.

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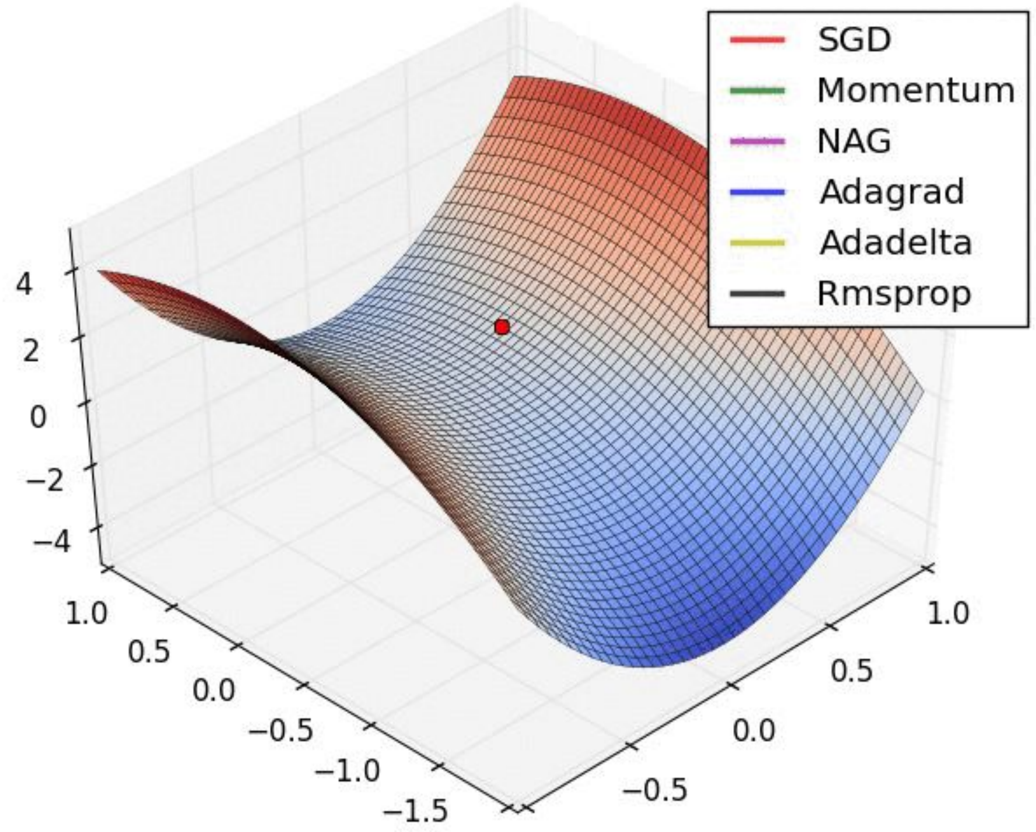
Demo

Adagrad & RMSprop



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$$\mathbf{w}_{t+1} = \mathbf{w}_t + \mathbf{m}_{t+1}$$

Momentum

$$\mathbf{v}_{t+1} = \beta \mathbf{v}_t + (1-\beta)\mathbf{g}_t^2$$
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RMSProp

$$\mathbf{v}_{t+1} = \beta \mathbf{v}_t + (1-\beta)\mathbf{g}_t^2$$

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \frac{\alpha}{\sqrt{\mathbf{v}_{t+1} + \epsilon}} \odot \mathbf{g}_t$$

RMSProp

ADAM
(**Ada**ptive **M**oment Estimate)

Cornell Bowers C·IS

$$\mathbf{v}_{t+1} = \beta_2 \mathbf{v}_t + (1 - \beta_2)\mathbf{g}_t^2$$

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \frac{\alpha}{\sqrt{\widehat{\mathbf{v}}_{t+1} + \epsilon}} \odot \widehat{\mathbf{m}}_{t+1}$$

$$\mathbf{v}_{t+1} = \beta \mathbf{v}_t + (1 - \beta)\mathbf{g}_t^2$$

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \frac{\alpha}{\sqrt{\mathbf{v}_{t+1} + \epsilon}} \odot \mathbf{g}_t$$

RMSProp

ADAM
(**Ada**ptive **M**oment Estimate)

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Momentum

$$\mathbf{v}_{t+1} = \beta \mathbf{v}_t + (1-\beta)\mathbf{g}_t^2$$
$$\mathbf{w}_{t+1} = \mathbf{w}_t - \frac{\alpha}{\sqrt{\mathbf{v}_{t+1} + \epsilon}} \odot \mathbf{g}_t$$

RMSProp

$$\mathbf{m}_{t+1} = \beta_1 \mathbf{m}_t + (1-\beta_1)\mathbf{g}_t$$
$$\mathbf{v}_{t+1} = \beta_2 \mathbf{v}_t + (1-\beta_2)\mathbf{g}_t^2$$
$$\mathbf{w}_{t+1} = \mathbf{w}_t - \frac{\alpha}{\sqrt{\widehat{\mathbf{v}}_{t+1} + \epsilon}} \odot \widehat{\mathbf{m}}_{t+1}$$

ADAM
(**Ada**ptive **M**oment Estimate)

$\mathbb{E}[\widehat{\mathbf{m}}_{t+1}]$

$\mathbb{E}[\widehat{\mathbf{v}}_{t+1}]$

$$\mathbf{m}_{t+1} = \beta_1 \mathbf{m}_t + (1-\beta_1)\mathbf{g}_t$$

$$\mathbf{v}_{t+1} = \beta_2 \mathbf{v}_t + (1-\beta_2)\mathbf{g}_t^2$$

$$\widehat{\mathbf{m}}_{t+1} = \frac{\mathbf{m}_{t+1}}{1-\beta_1^{t+1}}$$

$$\widehat{\mathbf{v}}_{t+1} = \frac{\mathbf{v}_{t+1}}{1-\beta_2^{t+1}}$$

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \frac{\alpha}{\sqrt{\widehat{\mathbf{v}}_{t+1}+\epsilon}} \odot \widehat{\mathbf{m}}_{t+1}$$

ADAM

(**Ada**ptive **M**oment Estimate)

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Optimizers Recap

- Gradient Descent
  - *Vanilla, costly, but for best convergence rate*
- Stochastic Gradient Descent
  - *Simple, lightweight*
- **Mini-batch SGD**
  - *balanced between SGD and GD*
  - ***1st choice for small, simple models***

- SGD w. Momentum
  - *Faster, capable to jump out local minimum*

- AdaGrad
- RMSProp
- **ADAM**
  - **JUST USE ADAM IF YOU DON'T KNOW WHAT TO USE IN DEEP LEARNING**