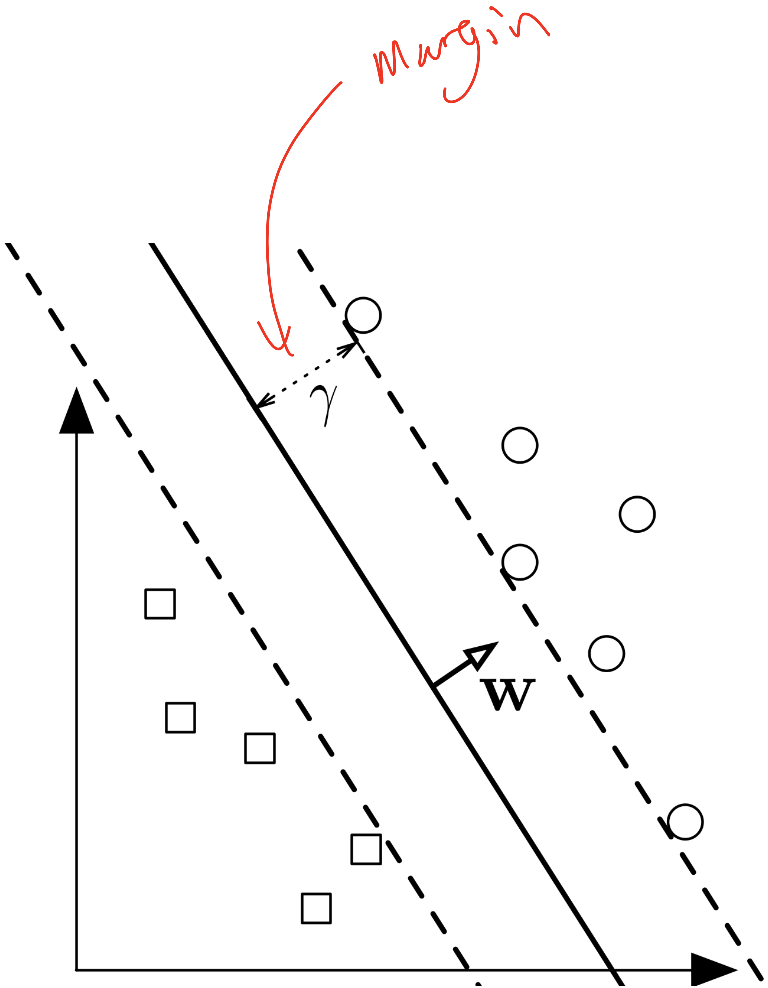
# Support Vector Machine (continue)

# Announcements

1. Prelim Conflict form is out and due next Tue

2. P4 is going to be out this afternoon (due after prelim)

# SVMs




**Goal of SVM**: find a hyperplane that (1) separates the data, (2) $\gamma(w, b)$ is maximized

# The SVM algorithm

$$\min_{w,b} \|w\|_2^2$$

$$\forall i: \ y_i(w^\top x_i + b) \geq 1$$

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Not only linearly separable, but also has functional margin no less than 1

$$y_i(w^\top x_i + b) > 0$$

# The SVM algorithm

Avoids "cheating" (i.e., scale $w, b$ up by large constant)

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Denote $(w, b)$ as the optimal solution:

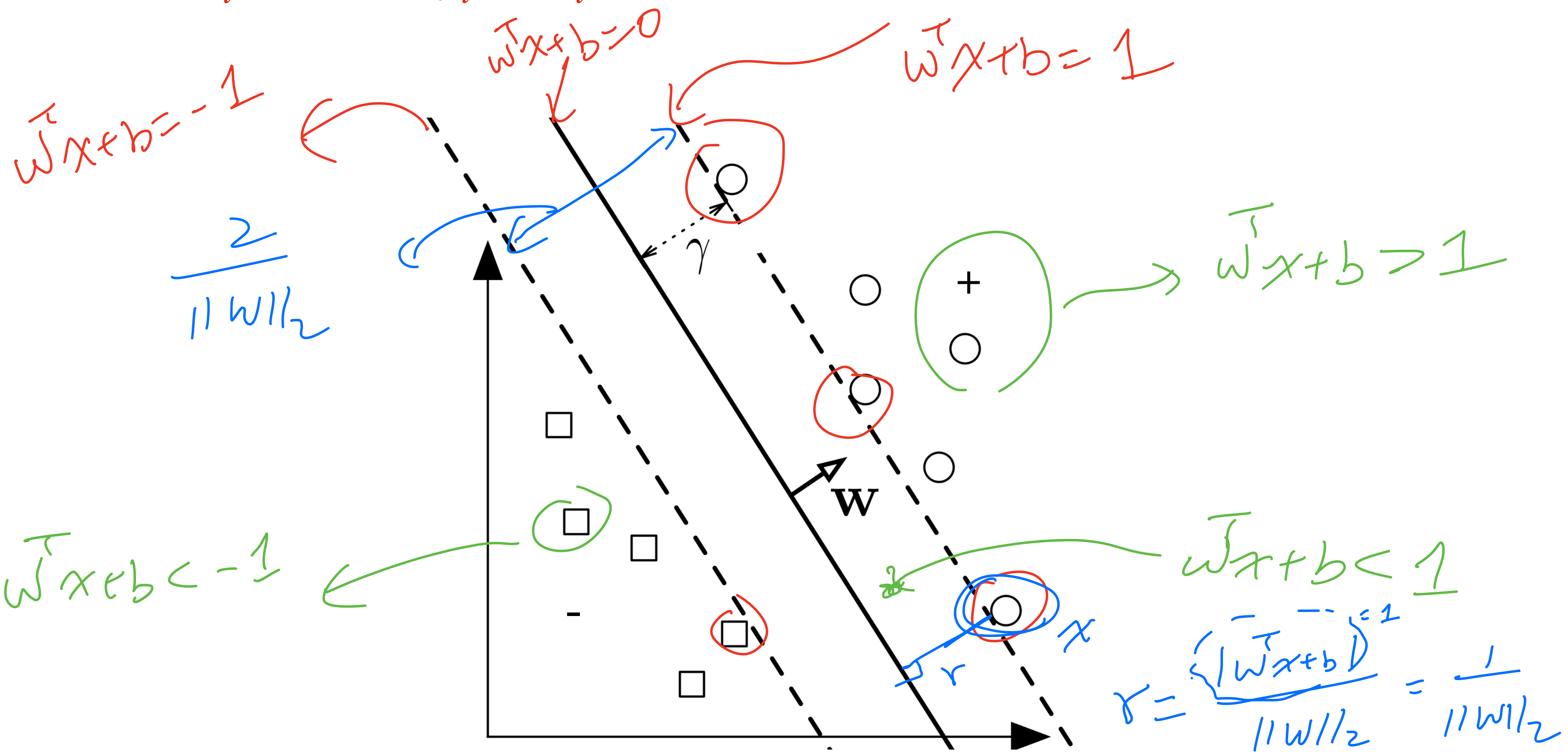
Q: will there be some $(x, y)$, such that $y(w^\top x + b) = 1$?

$\min_i y_i(w^T x_i + b) = c > 1$

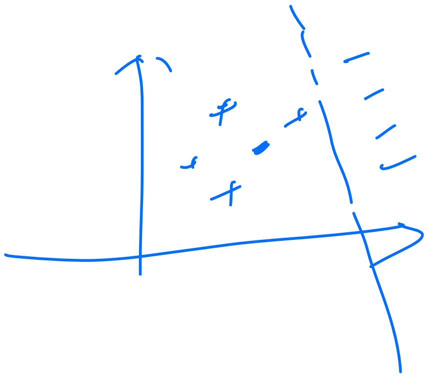
$w' = \frac{w}{c}$ $b' = \frac{b}{c}$

# Support Vectors

Points $x_i$ such that $y_i(w^\top x_i + b) = 1$ are called **support vectors**



# SVM for non-separable data



$$\min_{w,b} \|w\|_2^2 + c\sum_{i=1}^{n} \max\left\{0, 1 - y_i(w^\top x_i + b)\right\}$$

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$$\min_{w,b} \|w\|_2^2 + c \sum_{i=1}^{n} \boxed{\max\left\{0, 1 - y_i(w^\top x_i + b)\right\}}$$

Hinge loss

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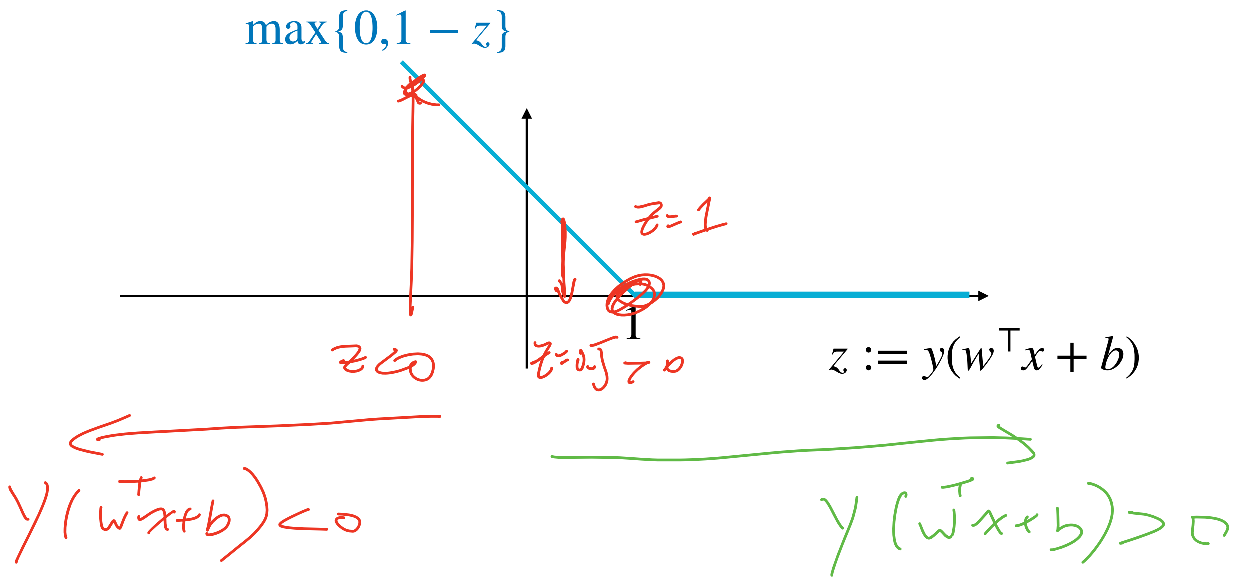

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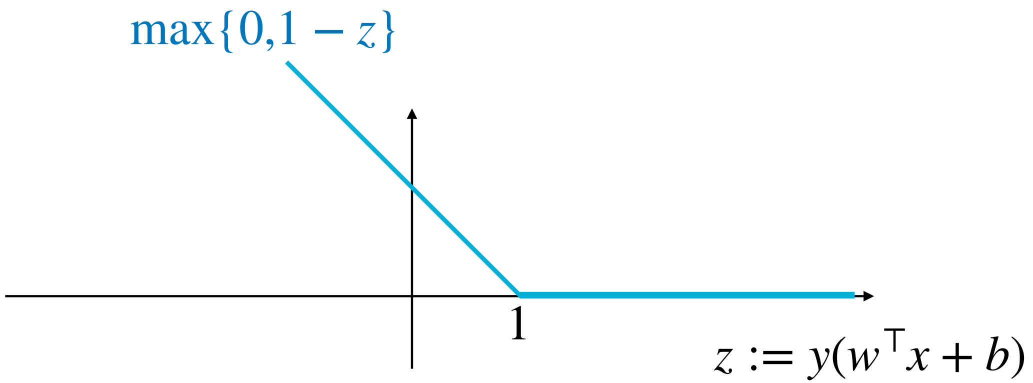


Hinge loss starts penalizing when functional margin falls below $1$

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When $c \to +\infty$:

forcing $y_i(w^\top x_i + b) \geq 1$ for as many data points as possible

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When $c \to 0^+$:

The solution $w \to \mathbf{0}$ (i.e., we do not care about hinge loss part)

$$\min_{w,b} \|w\|_2^2 + c \sum_{i=1}^{n} \max \left\{0, 1 - y_i(w^\top x_i + b)\right\}$$

width of the "street" $= \cancel{2/\|w\|_2}$

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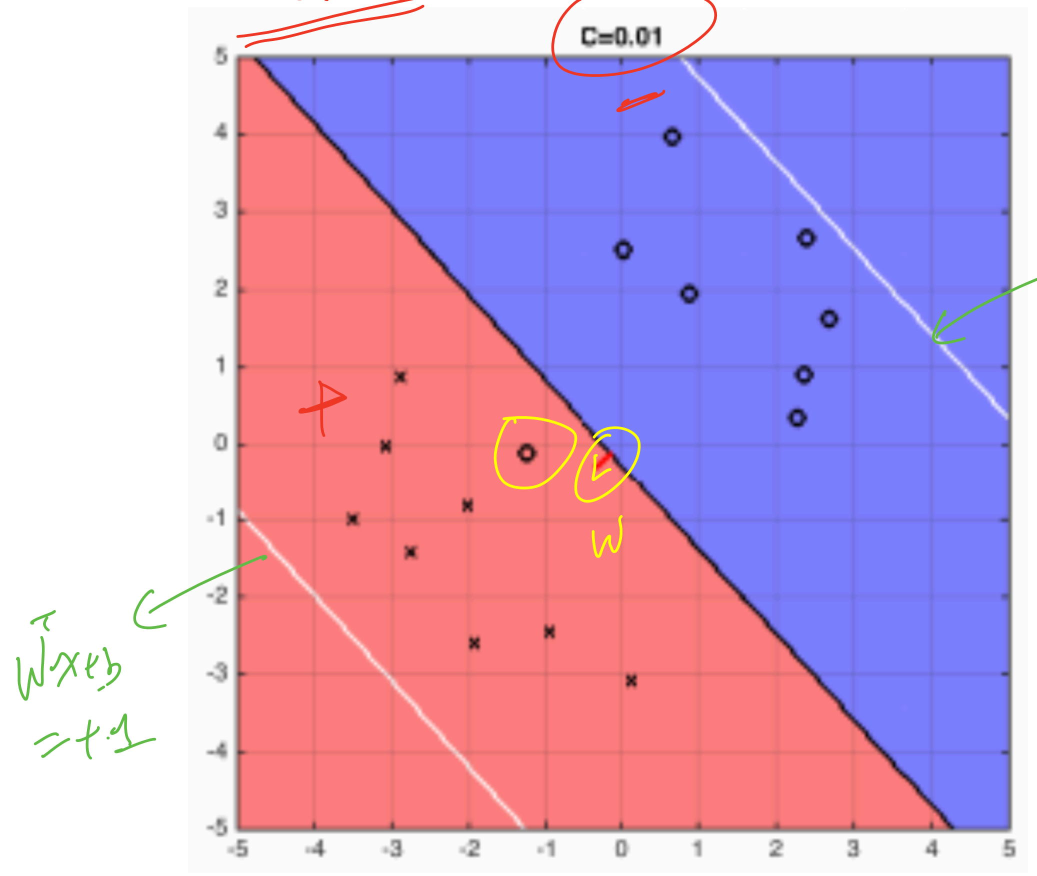
0.01



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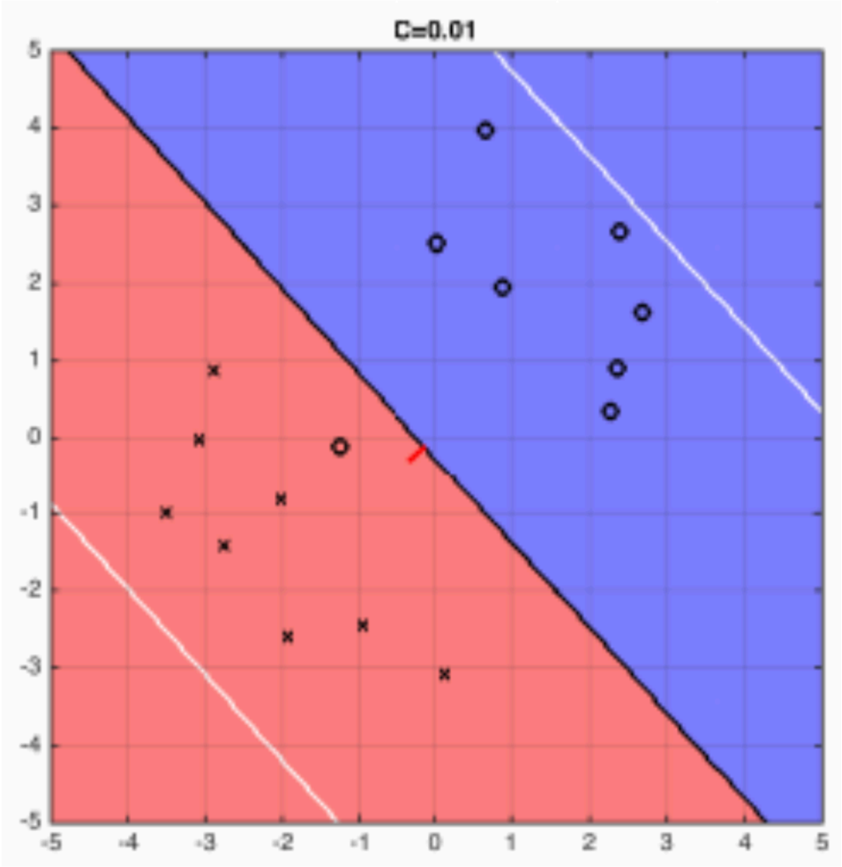


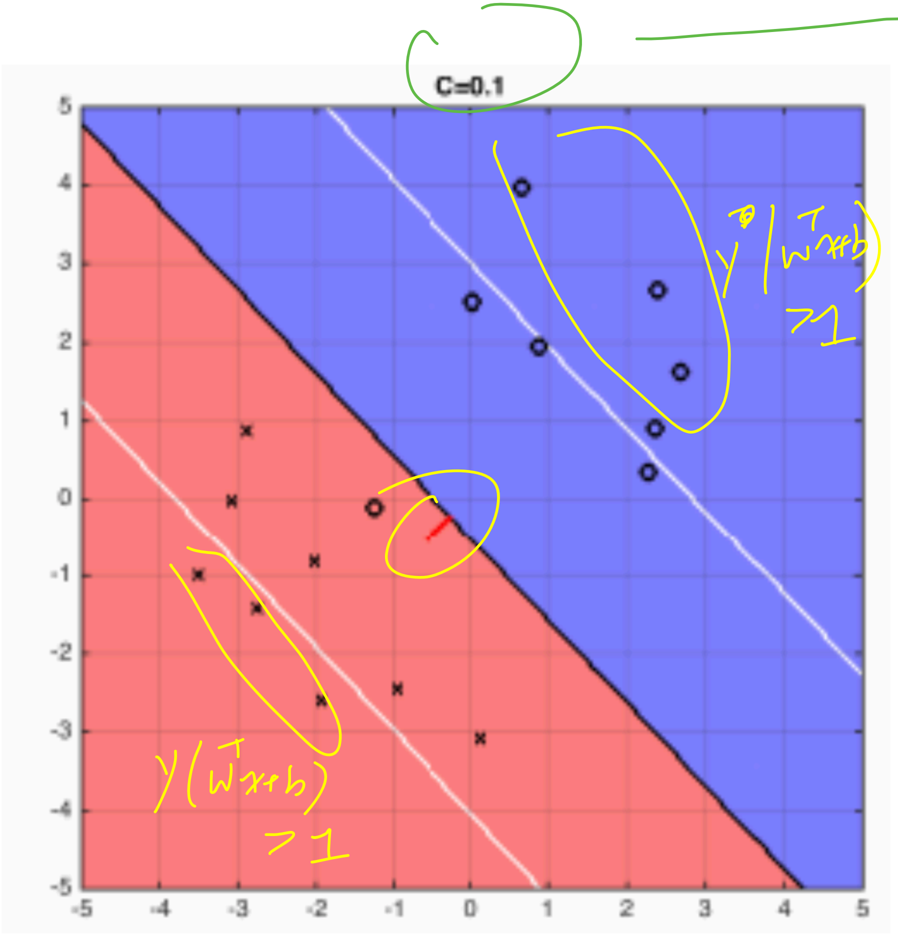

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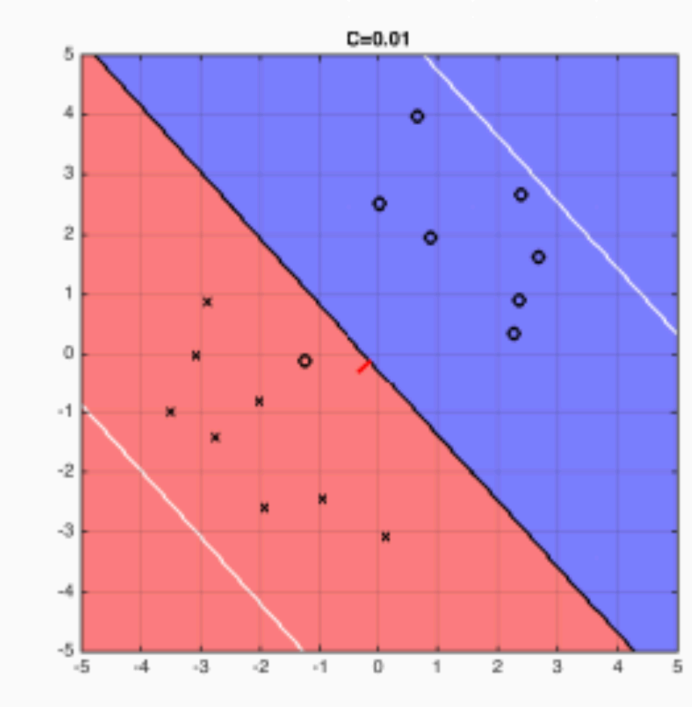

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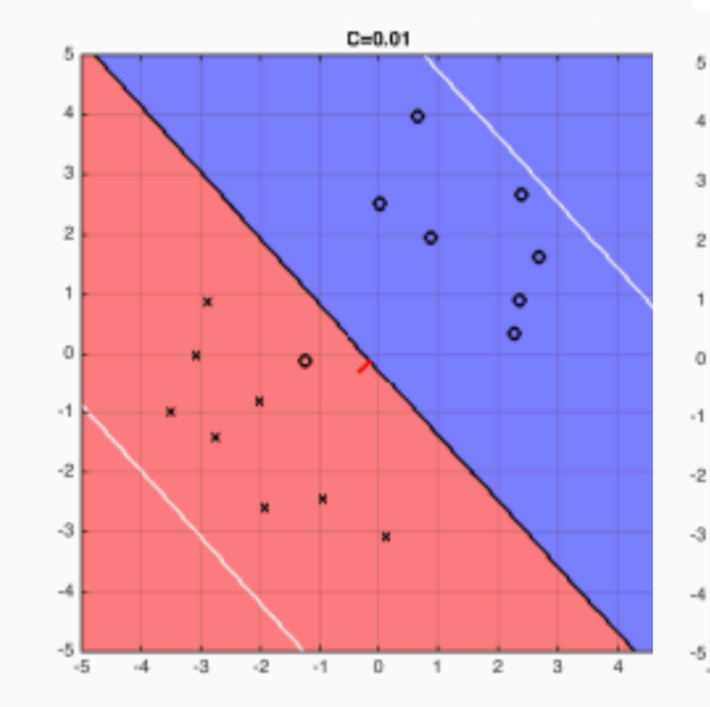


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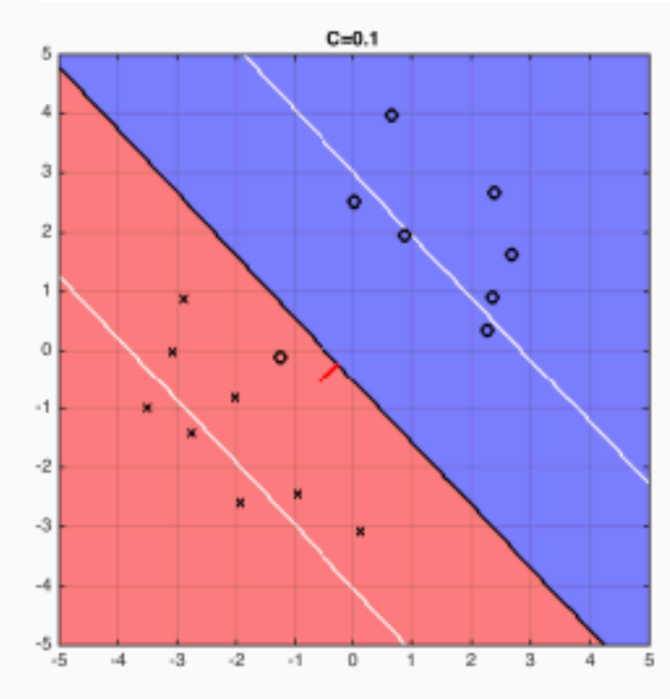


$$\min_{w,b} \|w\|_2^2 + c\sum_{i=1}^{n} \max\left\{0, 1 - y_i(w^\top x_i + b)\right\}$$

$w^T x + b = 0$

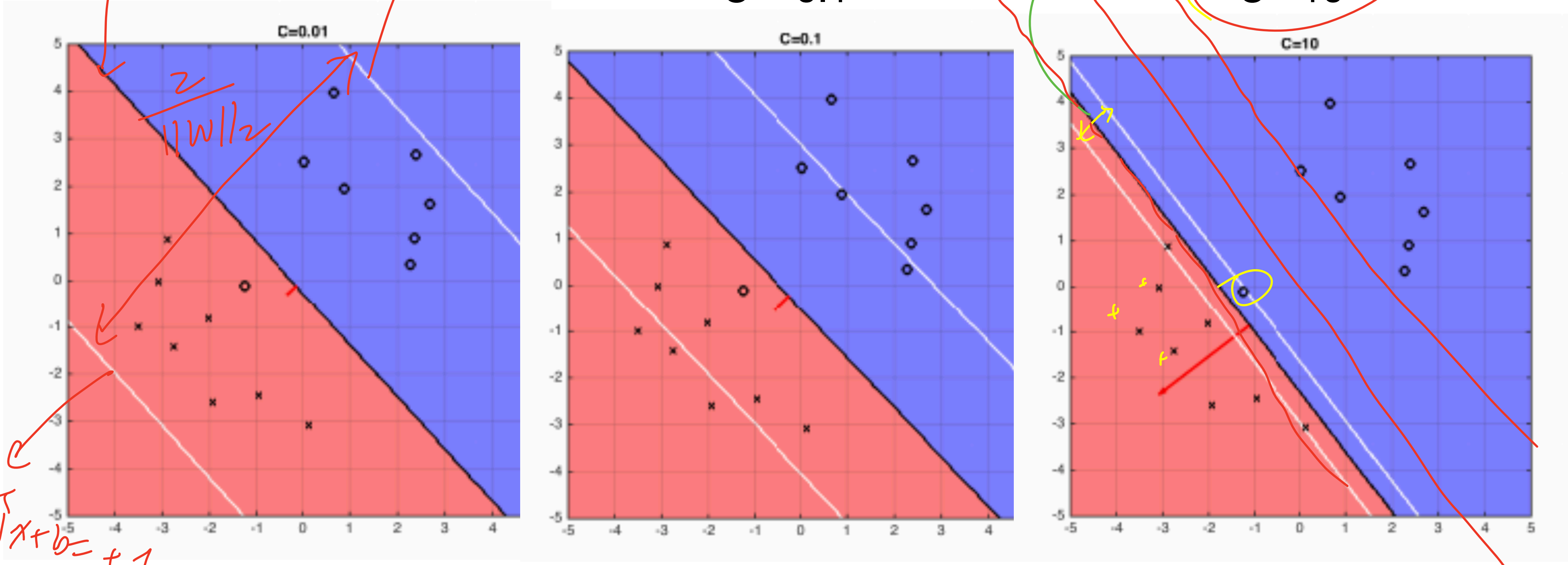
$w^T x + b = -1$

$\frac{1}{\|w\|_2}$

C = 0.01

C = 0.1

C =10




$w^T x + b = +1$

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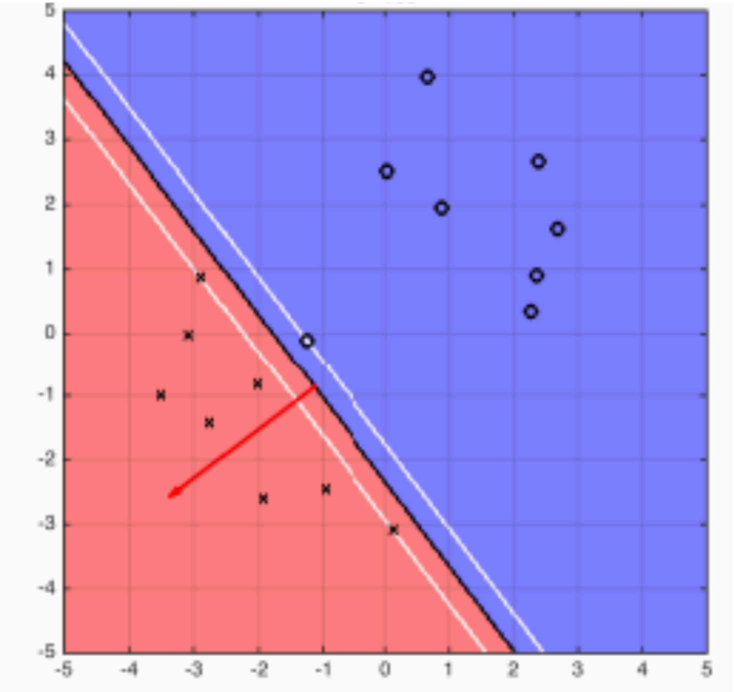

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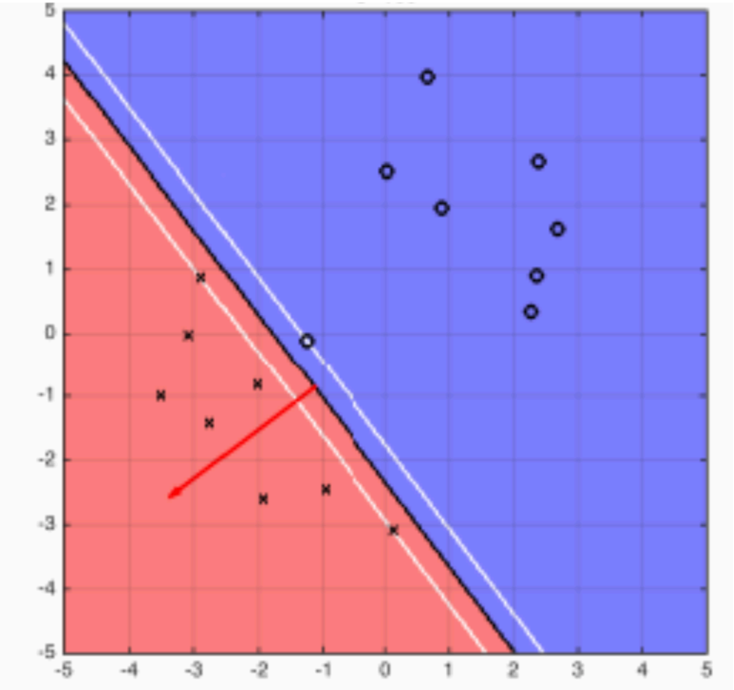


all examples have zero Hinge loss, but

$w$ has large norm

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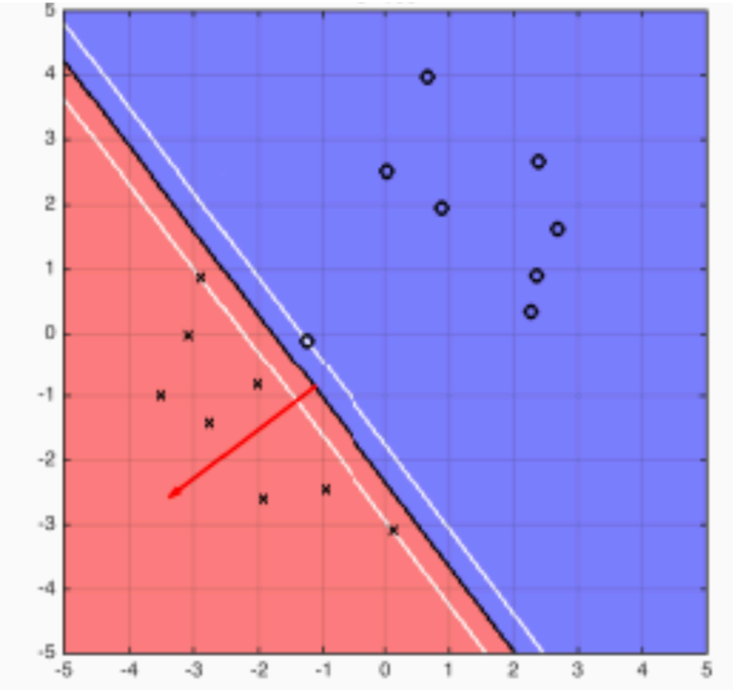


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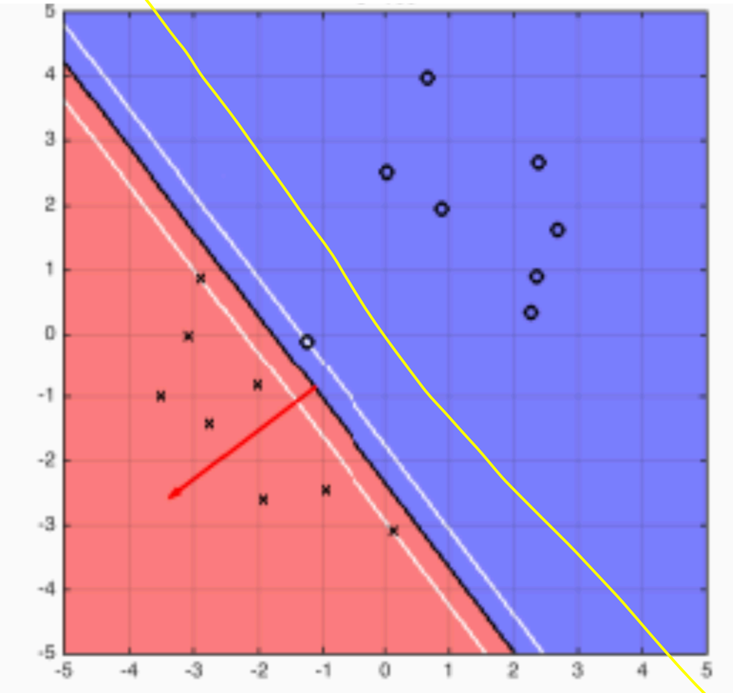

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Bad geometric margin but good functional margin (achieved by "cheating")

Potentially overfitting to the noise, not a good classifier in test time maybe

# Empirical Risk Minimization

# ERM

Recall the general supervised learning setting:

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Hypothesis $h : \mathscr{X} \to \mathbb{R}$, & hypothesis class $\mathscr{H} := \{h\} \subset \mathscr{X} \mapsto \mathbb{R}$

$\{+1, -1\} \leftarrow$ classification

$\mathbb{R} \leftarrow$ Regression

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Hypothesis $h : \mathscr{X} \to \mathbb{R}$, & hypothesis class $\mathscr{H} := \{h\} \subset \mathscr{X} \mapsto \mathbb{R}$

Loss function: $\ell(\underline{h(x)}, y)$

our prediction

# ERM

The ultimate objective function:

$$\arg\min_{h \in \mathcal{H}} \mathbb{E}_{x,y \sim P} \left[ \ell(h(x), y) \right]$$

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$n \to \infty$

Instead we have its **empirical** version

$$\arg\min_{h\in\mathcal{H}} \frac{1}{n}\sum_{i=1}^{n}\left[\ell(h(x_i), y_i)\right]$$

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Empirical risk / Empirical error

# The generalization error of ERM solution

$$\hat{h}_{ERM} := \arg\min_{h \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^{n} \left[\ell(h(x_i), y_i)\right]$$

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We often are interested in the true performance of $\hat{h}_{ERM}$:

$$\mathbb{E}_{\mathcal{D}} \left[ \mathbb{E}_{x,y \sim P} \underbrace{\ell(\hat{h}_{ERM}(x), y)}_{} \right]$$

$\mathcal{D}$

$\hat{h}_{ERM}$ is dependent on $\mathcal{D}$

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# The generalization error of ERM solution

$x_i, y_i \sim P$

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We often are interested in the true performance of $\hat{h}_{ERM}$:

Training

$$\mathbb{E}_{\mathcal{D}} \left[ \mathbb{E}_{x,y \sim P} \ell(\hat{h}_{ERM}(x), y) \right]$$

Testing

Note $\hat{h}_{ERM}$ is a random quantity as it depends on data $\mathcal{D}$

e.g., In LR: $\hat{w} = (XX^\top)^{-1} XY$.

# The generalization error of ERM solution

Ideally, we want the true loss of ERM to be small:

$$\underbrace{\mathbb{E}_{\mathcal{D}}\left[\mathbb{E}_{x,y\sim P}\ell(\hat{h}_{ERM}(x),y)\right]}_{\text{performance of ERM}} \approx \underbrace{\min_{h\in\mathcal{H}}\mathbb{E}_{x,y\sim P}\ell(h(x),y)}$$

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The Minimum expected loss we could get if we knew $P$

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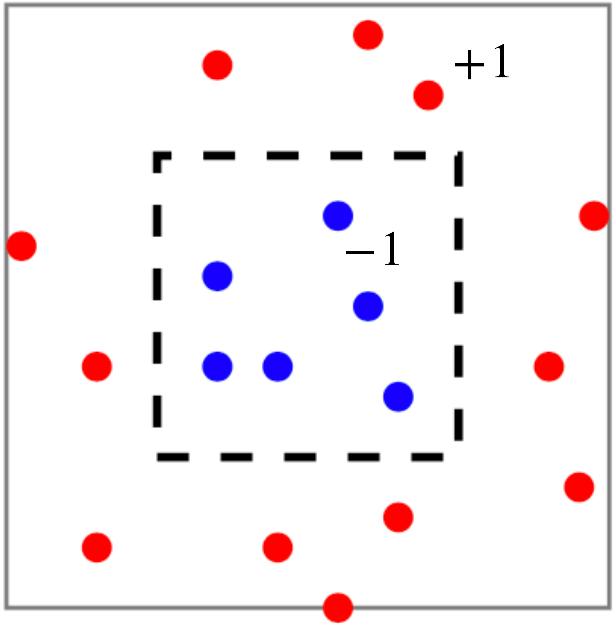
The Minimum expected loss we could get if we knew $P$

However, this may not hold if we are not careful about designing $\mathcal{H}$

# Example:

$P$: $x$ uniformly distribution over the square;
Label: blue if inside the smaller square, else red



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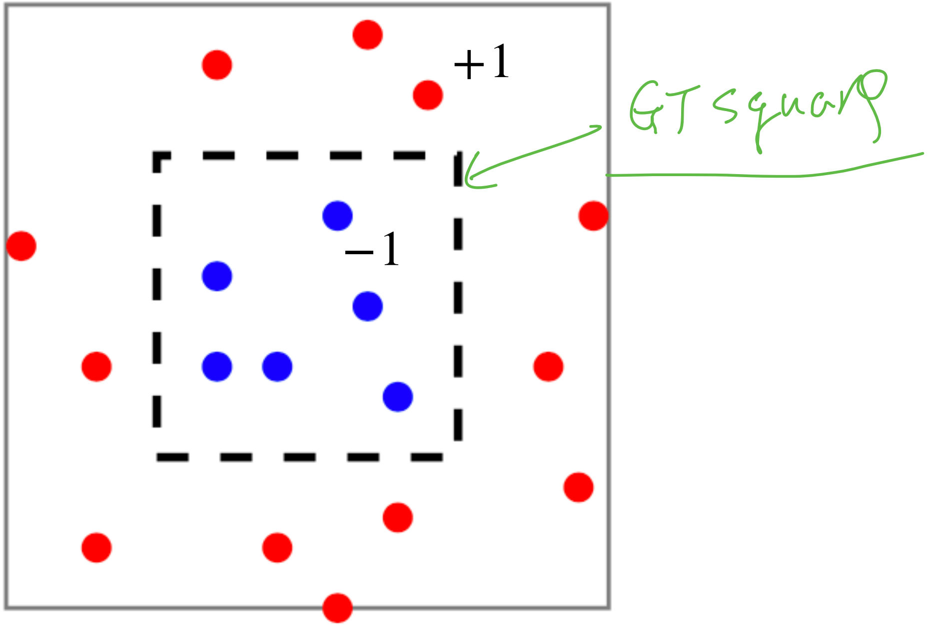


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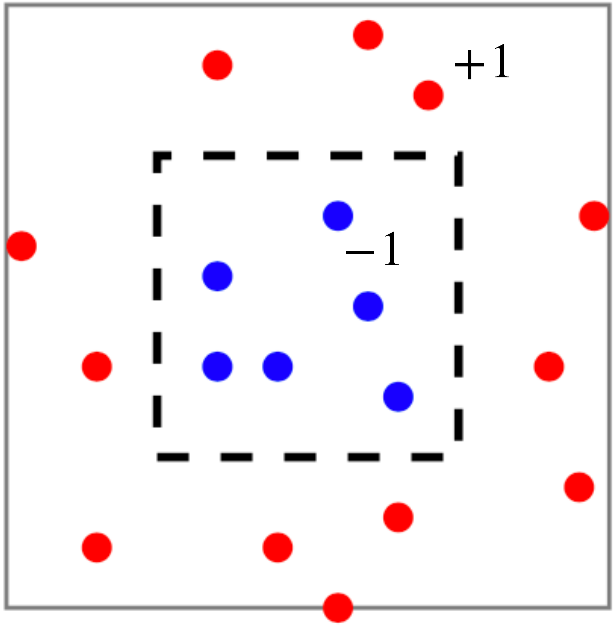


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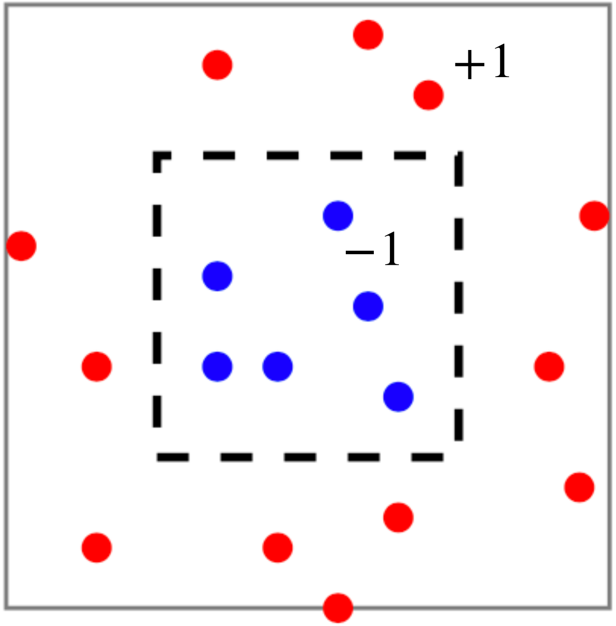


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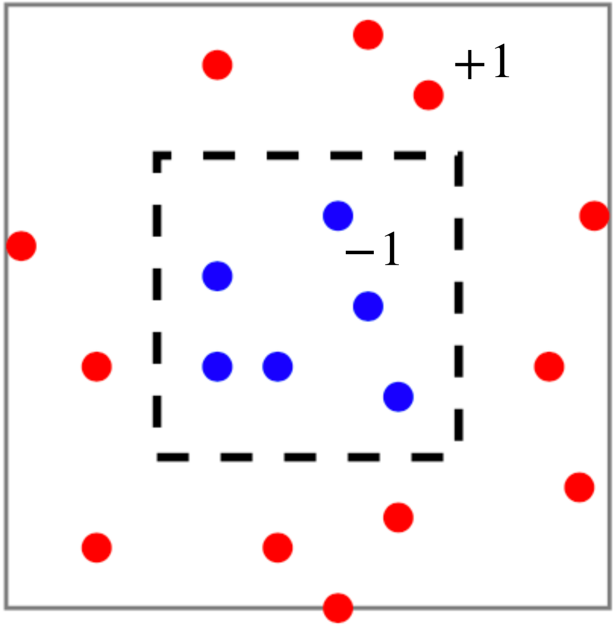


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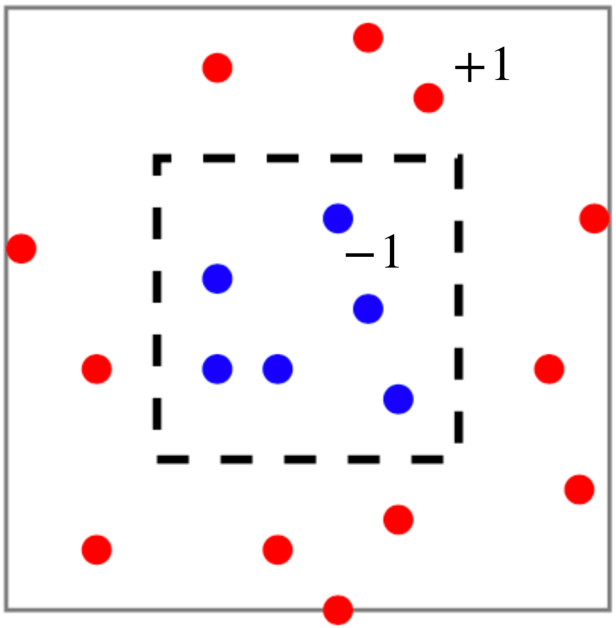


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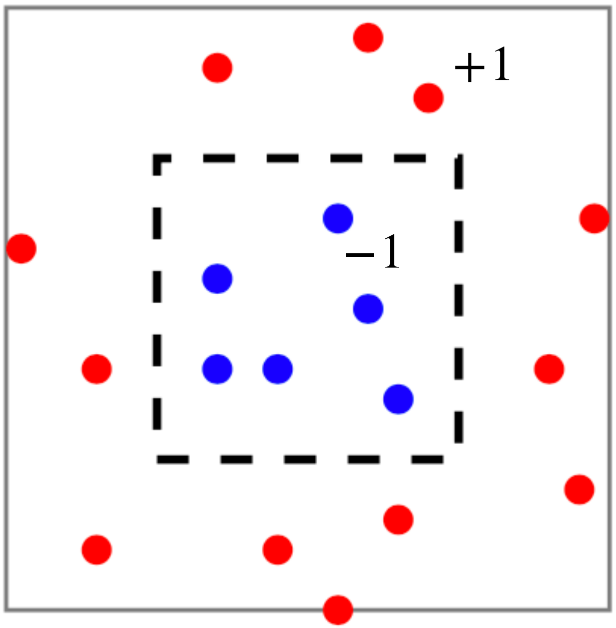


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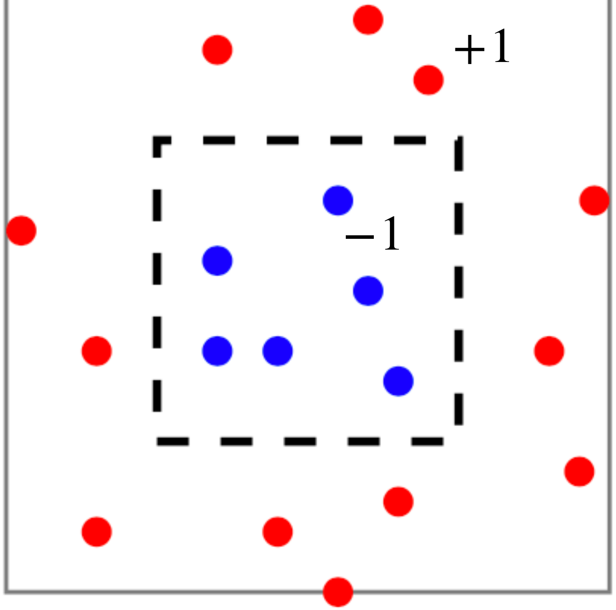


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Q: But what's the true expected error of this $\hat{h}$?

A: area of smaller box / total area

# ERM with inductive bias

A common solution is to restrict the search space (i.e., hypothesis class)

$$\hat{h}_{ERM} := \arg\min_{h \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^{n} \left[\ell(h(x_i), y_i)\right]$$

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By restricting to $\mathcal{H}$, we bias towards solutions from $\mathcal{H}$

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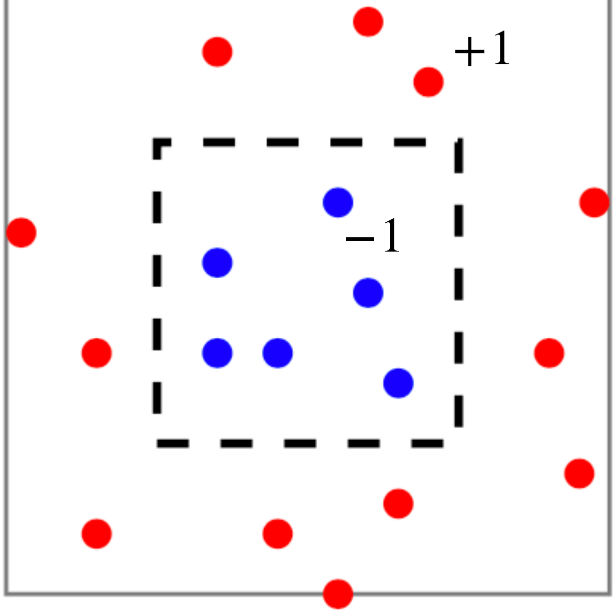


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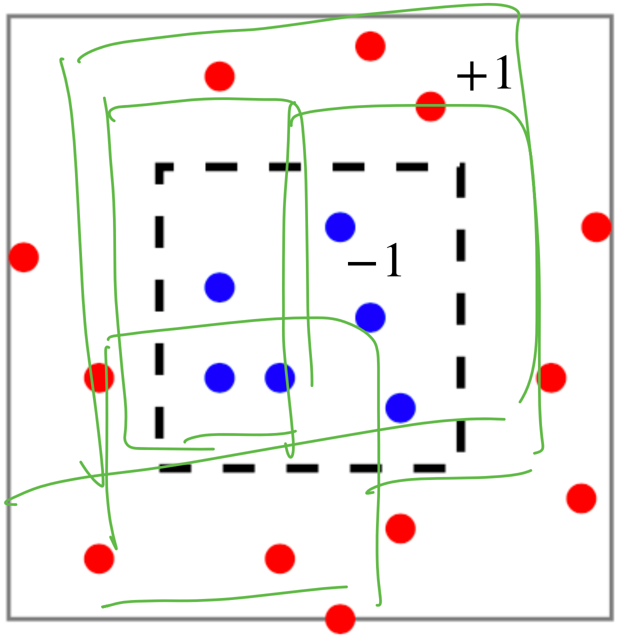


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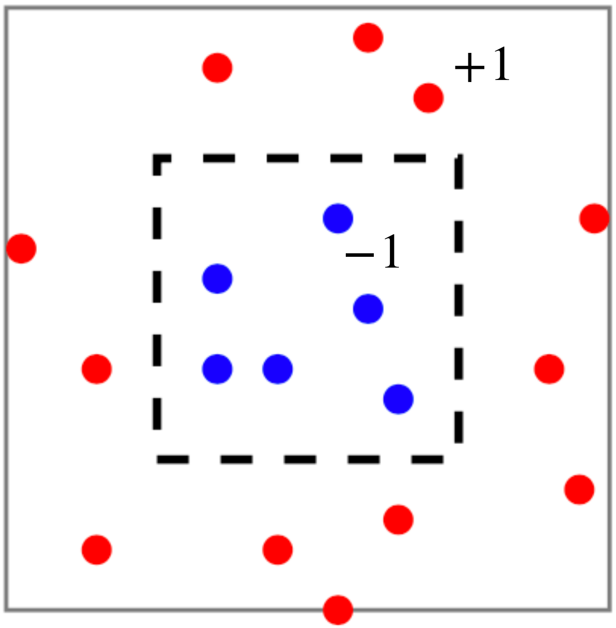


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$$\mathbb{E}_{\mathscr{D}}\left[\mathbb{E}_{x,y\sim P}\ell(\hat{h}_{ERM}(x), y)\right]$$

$$\leq \min_{h\in\mathscr{H}} \mathbb{E}_{x,y\sim P}\ell(h(x), y) + O(1/\sqrt{n})$$

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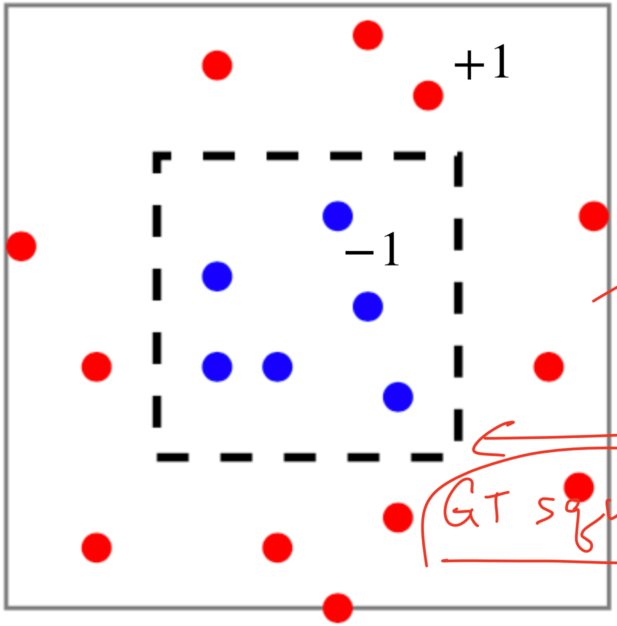


GT square

Unrestricted hypothesis class did not work;

However, if we restrict $\mathscr{H}$ to contains ALL axis-aligned rectangles, then ERM will succeed, i.e.,

$$\mathbb{E}_{\mathscr{D}}\left[\mathbb{E}_{x,y\sim P}\ell(\hat{h}_{ERM}(x),y)\right]$$

$$\le \min_{h\in\mathscr{H}} \mathbb{E}_{x,y\sim P}\ell(h(x),y) + O(1/\sqrt{n})$$

$$\le O(1/\sqrt{n}) \qquad = 0$$

# Example:

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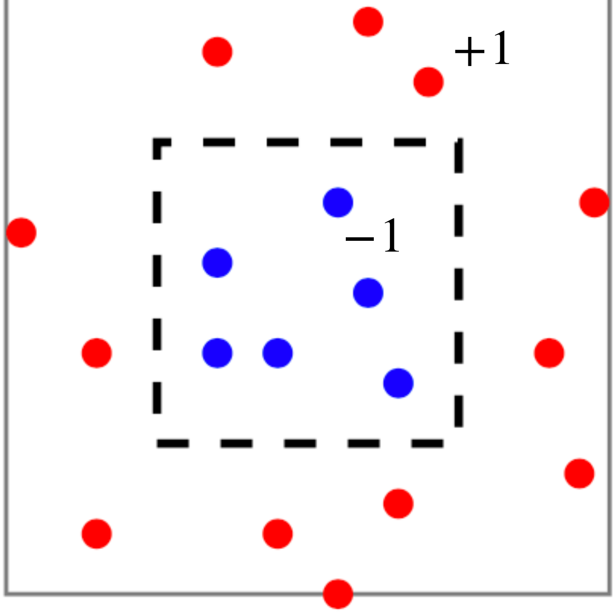


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$$\leq O(1/\sqrt{n})$$

(Exact proof out of the scope of this class — see CS 4783/5783)

# Summary

ERM with unrestricted hypothesis class could fail (i.e., overfitting)

To guarantee small test error, we need to restrict $\mathscr{H}$

# After Prelim

We will continue from ERM:

Examples of loss functions,
ways to restrict the hypothesis classes,
why that really matters in ML (theory and practice)