Logistic Regression & convex optimization

Announcements:

This week we will release P3 and HW3

Recap on Naive Bayes

NB is a **generative model** which models $P(x, y)$

$$P(y \mid x) \propto P(y)P(x \mid y) = P(y)\prod_{i=1}^{d} P(x[i] \mid y)$$

Conditional independent
assumption given label

Perceptron VS Gaussian Naive Bayes

Today

Logistic regression — a ***discriminative learning*** approach that directly models $P(y \mid x)$ for classification

Outline for today

1. Logistic Regression

2. Convex optimization

3. Gradient Descent

Logistic Regression

Setting: binary classification $\mathscr{D} = \{x_i, y_i\}_{i=1}^n$, $(x_i, y_i) \sim P$,
$x_i \in \mathbb{R}^d, y_i \in \{-1, +1\}$

(Note, we always assume x contains a constant 1)

Logistic regression **directly models** $P(y \mid x)$

$$P(y \mid x) = \frac{1}{1 + \exp\left(-y(x^\top w^\star)\right)}$$

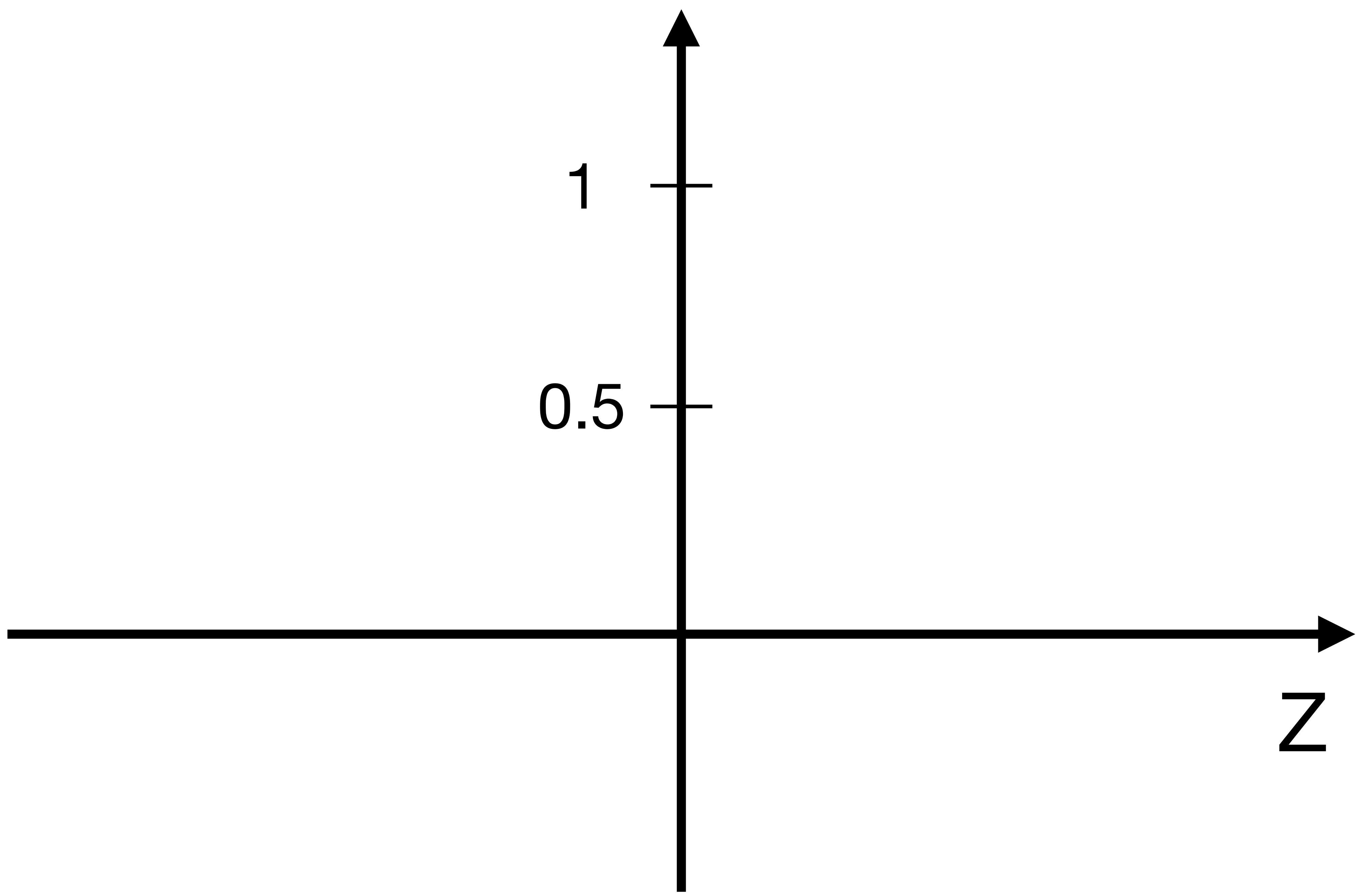
Logistic Regression

Logistic regression assumes:

$$P(y\,|\,x) = \frac{1}{1 + \exp\left(-y(x^\top w^\star)\right)}$$

The model assigns higher prob to
$y = \text{sign}(x^\top w^\star)$

Draw the Sigmoid function $1/(1 + \exp(-Z))$



Logistic Regression

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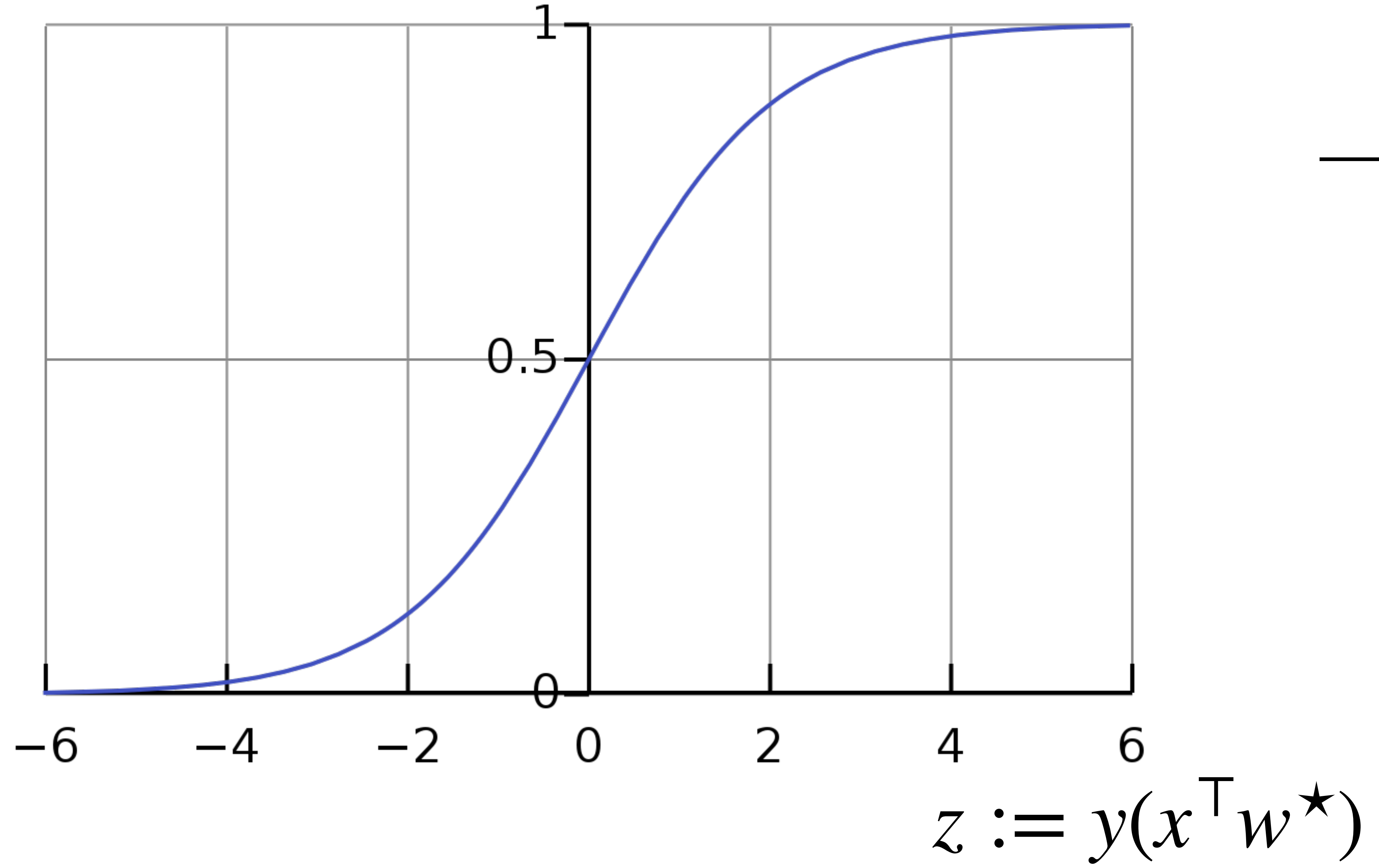
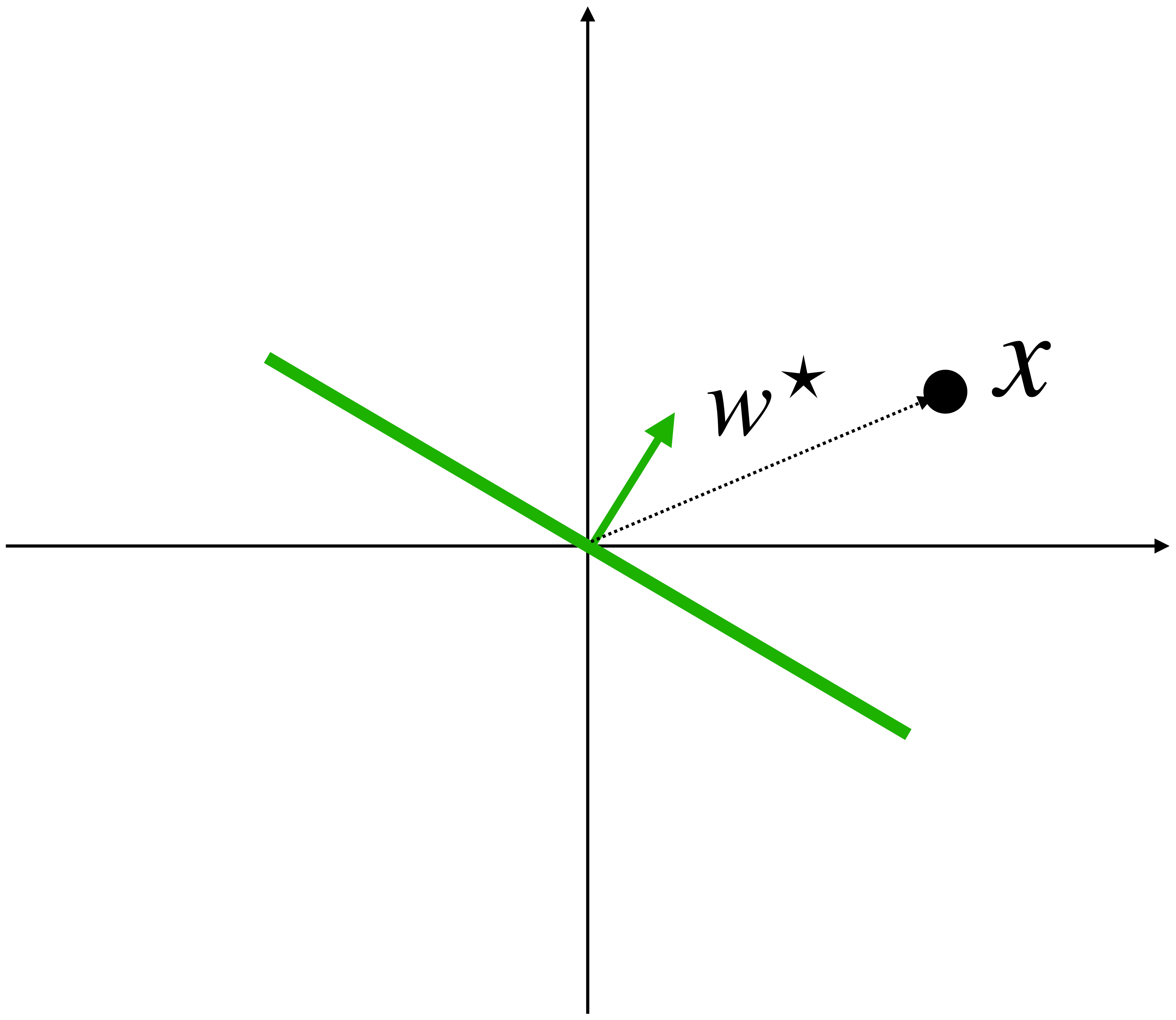




Learn via MLE

Recall we have data $\mathcal{D} = \{x_i, y_i\}_{i=1}^n$

$$\arg\max_w P(\mathcal{D} \mid w) = \arg\max_w P\left(\{y_i\}_{i=1}^n \mid \{x_i\}_{i=1}^n ; w\right)$$

$$= \arg\max_w \prod_{i=1}^n P\left(y_i \mid x_i ; w\right)$$

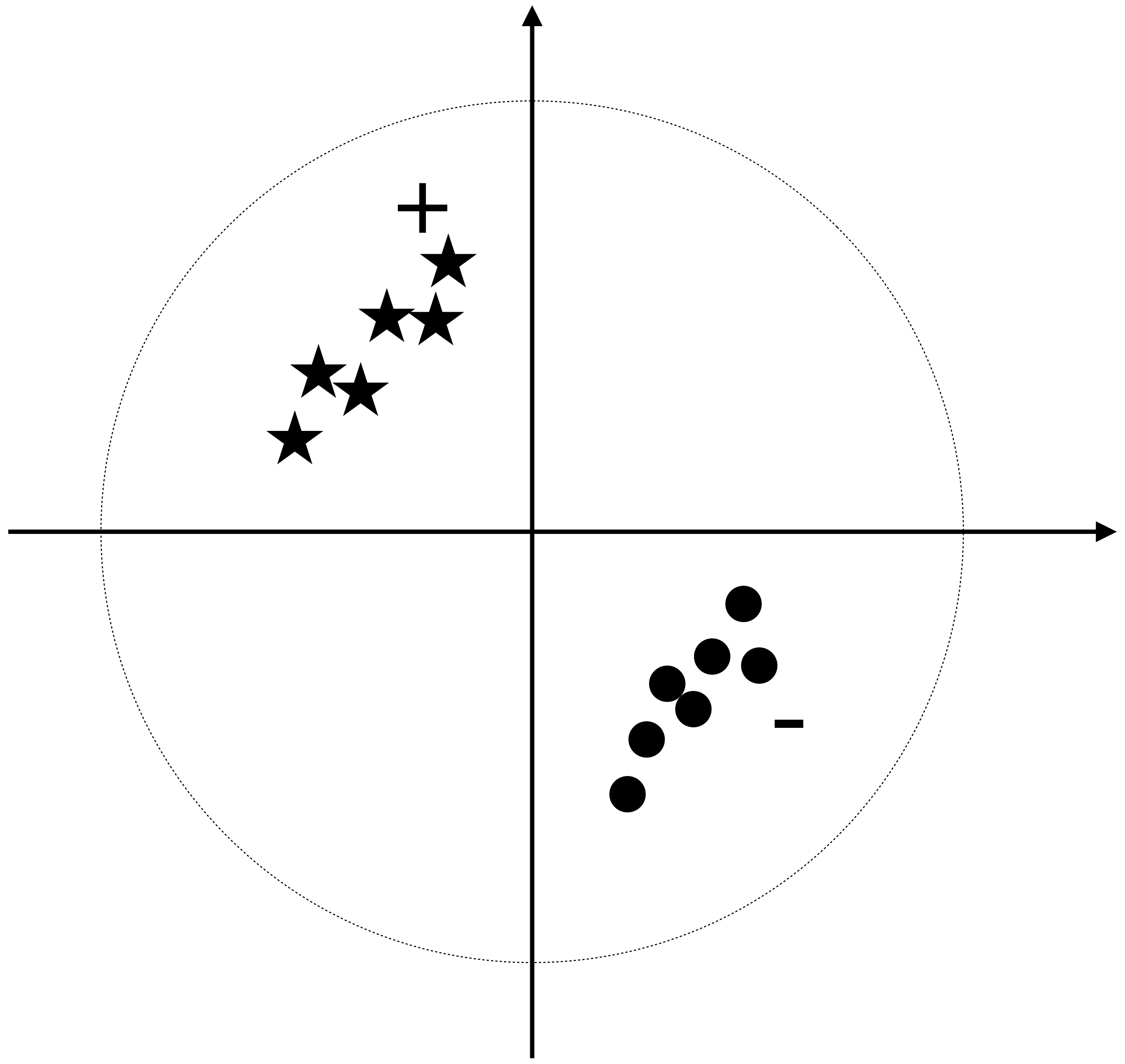
Plug in logistic assumption and add log:

$$\arg\max_w \sum_{i=1}^n - \ln\left[1 + \exp\left(-y_i (w^\top x_i)\right)\right]$$

Learn via MLE

$$\hat{w}_{mle} := \arg\max_{w} \sum_{i=1}^{n} \ln\left[\frac{1}{1+\exp\left(-y_i(w^\top x_i)\right)}\right]$$

Intuitively, $\hat{w}_{mle}$ tries to explain the label:



Q: for $y_i = +1$, what we should expect from $\hat{w}_{mle}^\top x_i$?

Q: for $y_i = -1$, what we should expect from $\hat{w}_{mle}^\top x_i$?

Learn via MAP

$$P(w \mid \mathscr{D}) \propto P(w)P(\mathscr{D} \mid w)$$

We use Gaussian prior, i.e., $P(w) = \mathcal{N}(0, \sigma^2 I)$

$$\arg\max_w \ln \left(P(w) \prod_{i=1}^{n} P(y_i \mid x_i, w) \right) = \arg\max_w \ln P(w) + \sum_{i=1}^{n} \ln P(y_i \mid x_i, w)$$

$$= \arg\min_w \left(\sum_{i=1}^{n} \ln \left(1 + \exp(-y_i(w^\top x_i))\right) + \frac{\|w\|_2^2}{2\sigma^2} \right)$$

Comparison to Navie Bayes

1. Logistic regression does not model $P(x|y)$

2. Gaussian NB leads a linear classifier in the form of

$$P(y|x) = 1/(1 + \exp(w^\top x))$$

Gaussian NB is a special case of logistic regression

Outline for today

1. Logistic Regression

2. Convex optimization

3. Gradient Descent

We needs to solve the optimization problem

$$\hat{w} := \arg\min_{w} \underbrace{\sum_{i=1}^{n} \ln\left[1 + \exp\left(-y_i(w^\top x_i)\right)\right] + \lambda\|w\|_2^2}_{:=\ell(w)}$$

There is no closed-form solution for the minimizer; luckily, $\ell(w)$ is convex

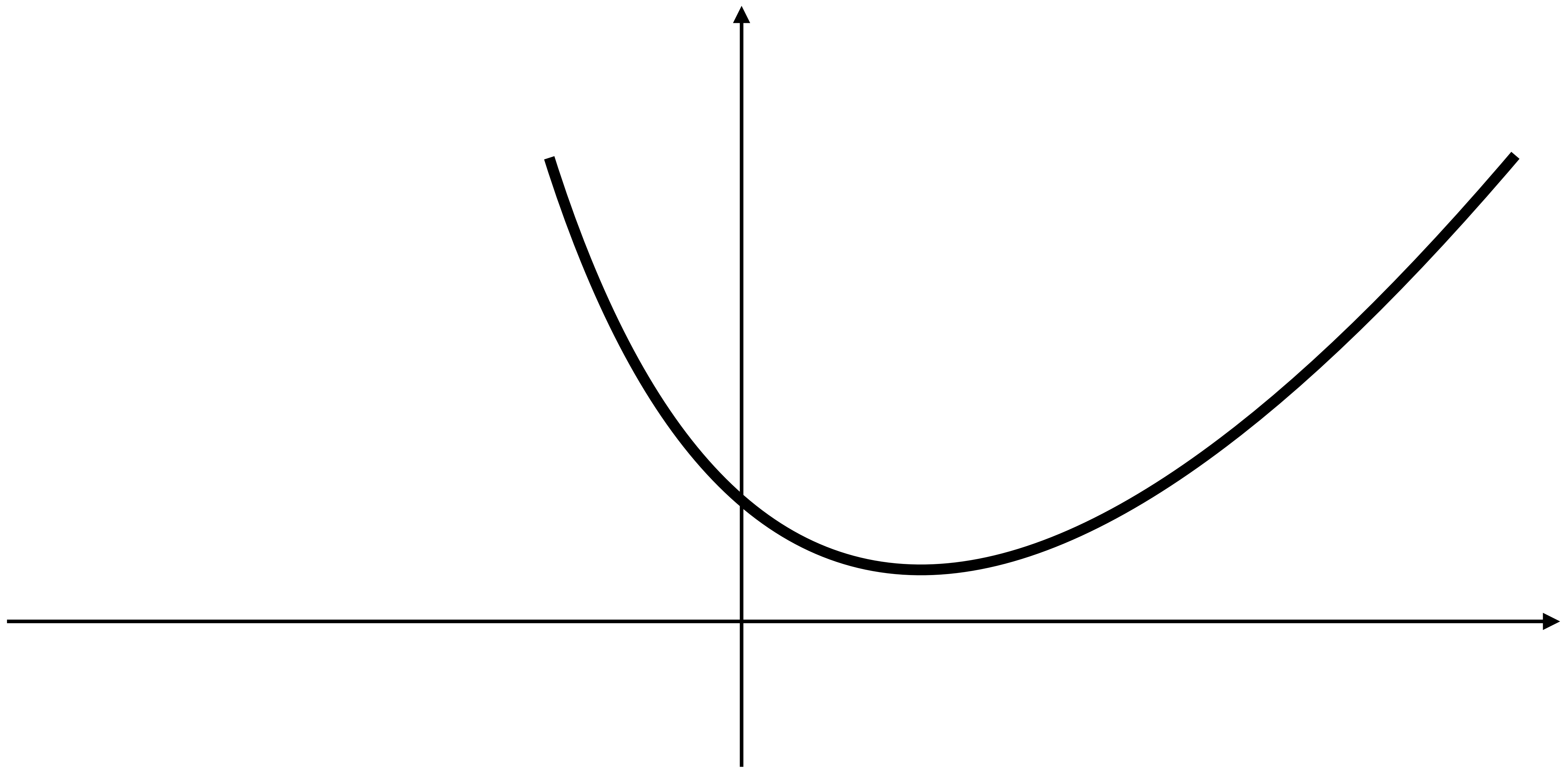
We will find an approximate minimizer via **gradient descent**

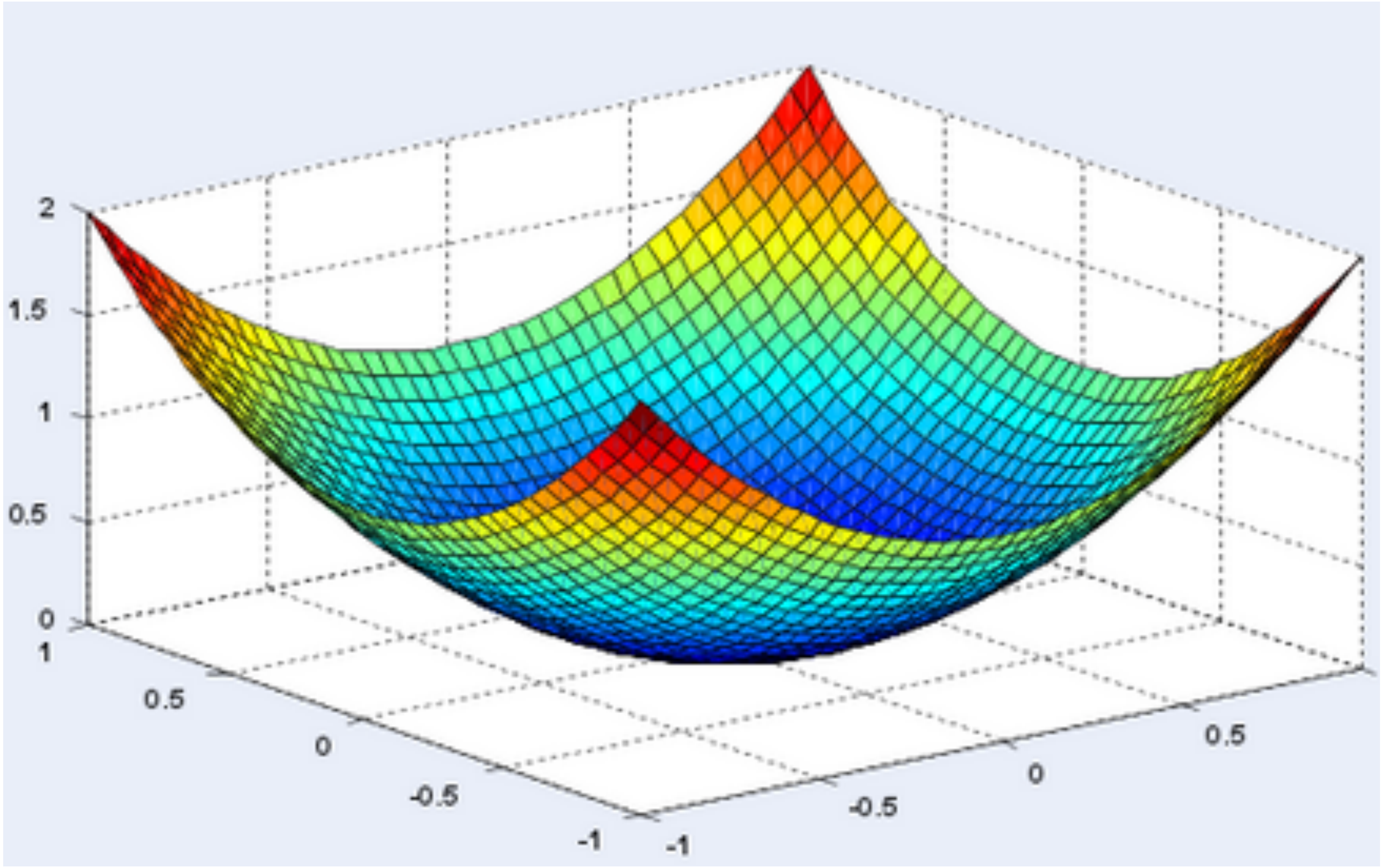
Setup for Optimization

We consider minimizing a (convex) function $\arg\min_{w} \ell(w)$

Def of convexity:

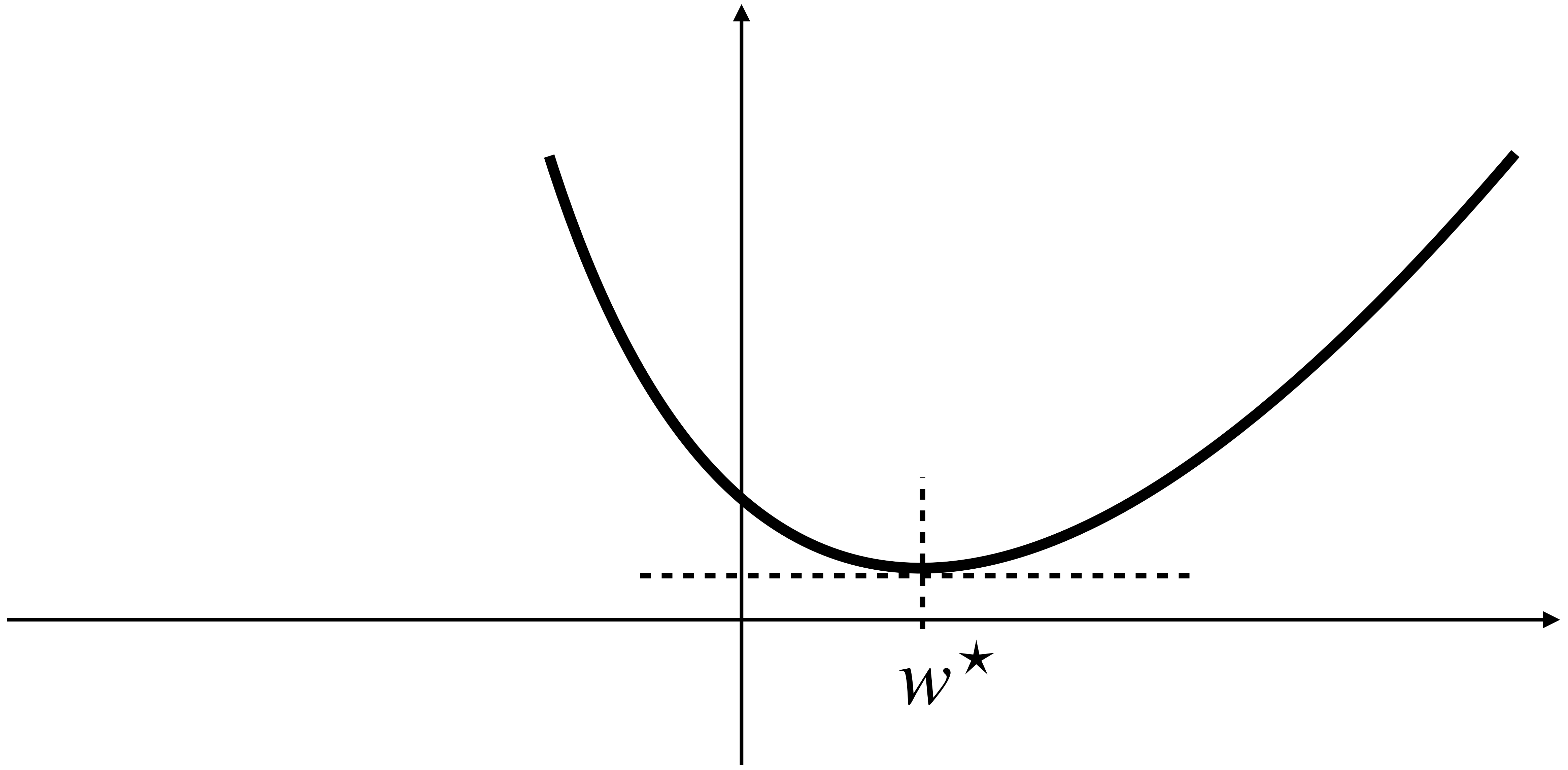
$$\forall (x, x'), \alpha \in [0,1],\ \ell(\alpha x + (1-\alpha)x') \leq \alpha \ell(x) + (1-\alpha)\ell(x')$$





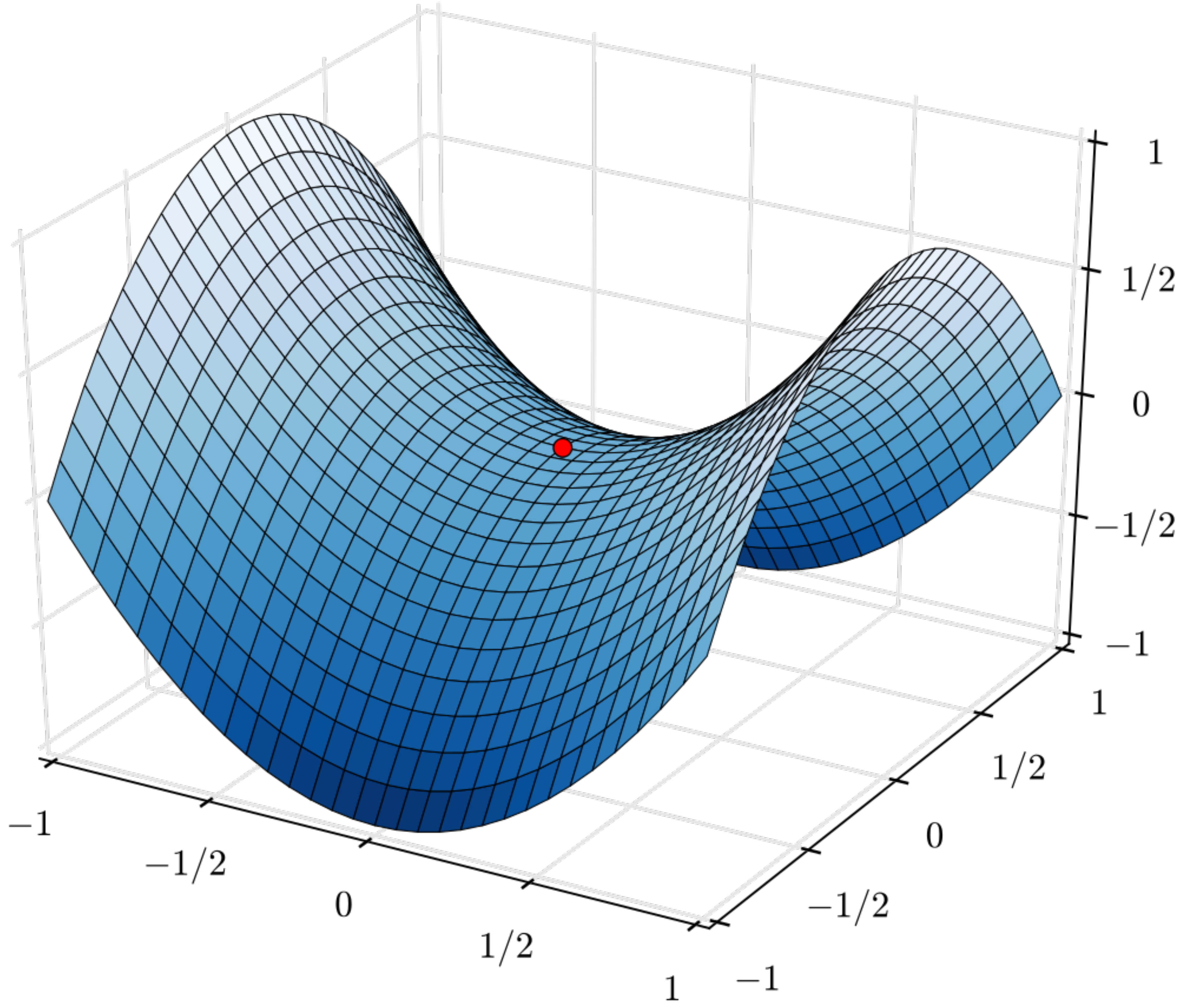
Global minimizer of a convex function

A convex function has global minimizer which has gradient equal to 0



Examples of non-convex functions

Saddle point ($\ell(x, y) = x^2 - y^2$)



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The Gradient Descent algorithm

Goal: minimize $\ell(w)$

Initialize $w^0 \in \mathbb{R}^d$

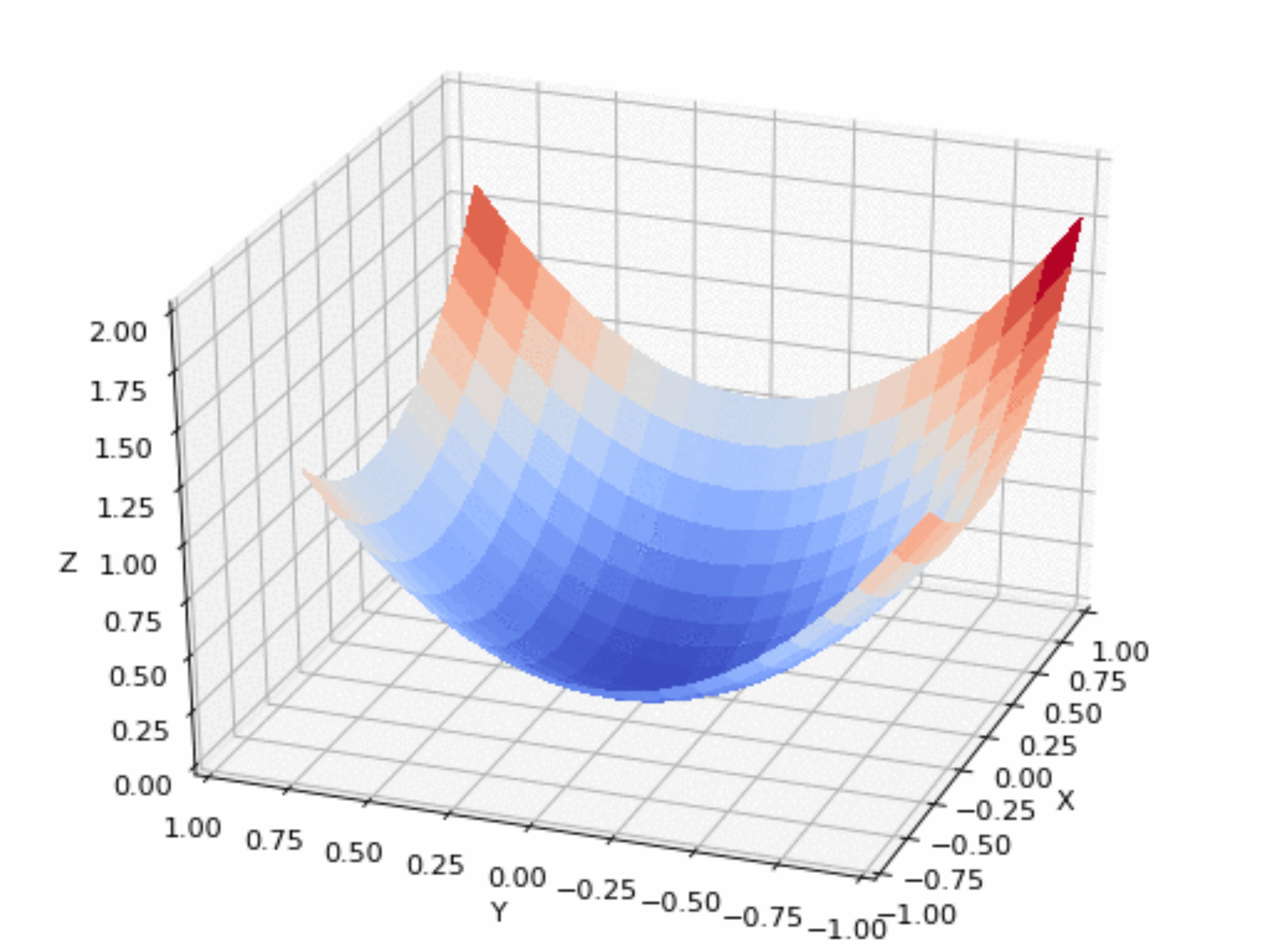
Iterate until convergence:

1. Compute gradient $g^t = \nabla \ell(w) \,|_{w=w_t}$
2. Update (GD): $w^{t+1} = w^t - \eta g^t$

η: learning rate

The Gradient Descent demo

$$\min_{x,y}(x^2 + y^2)$$



Informal proof for GD convergence

First-order Taylor expansion: for infinitesimally small δ (i.e., $\delta \to 0$), we have

$$\ell(w - \delta) = \ell(w) - \nabla \ell(w)^\top \delta$$

Substitute $\delta = \eta \nabla \ell(w)$, with $\eta \to 0^+$

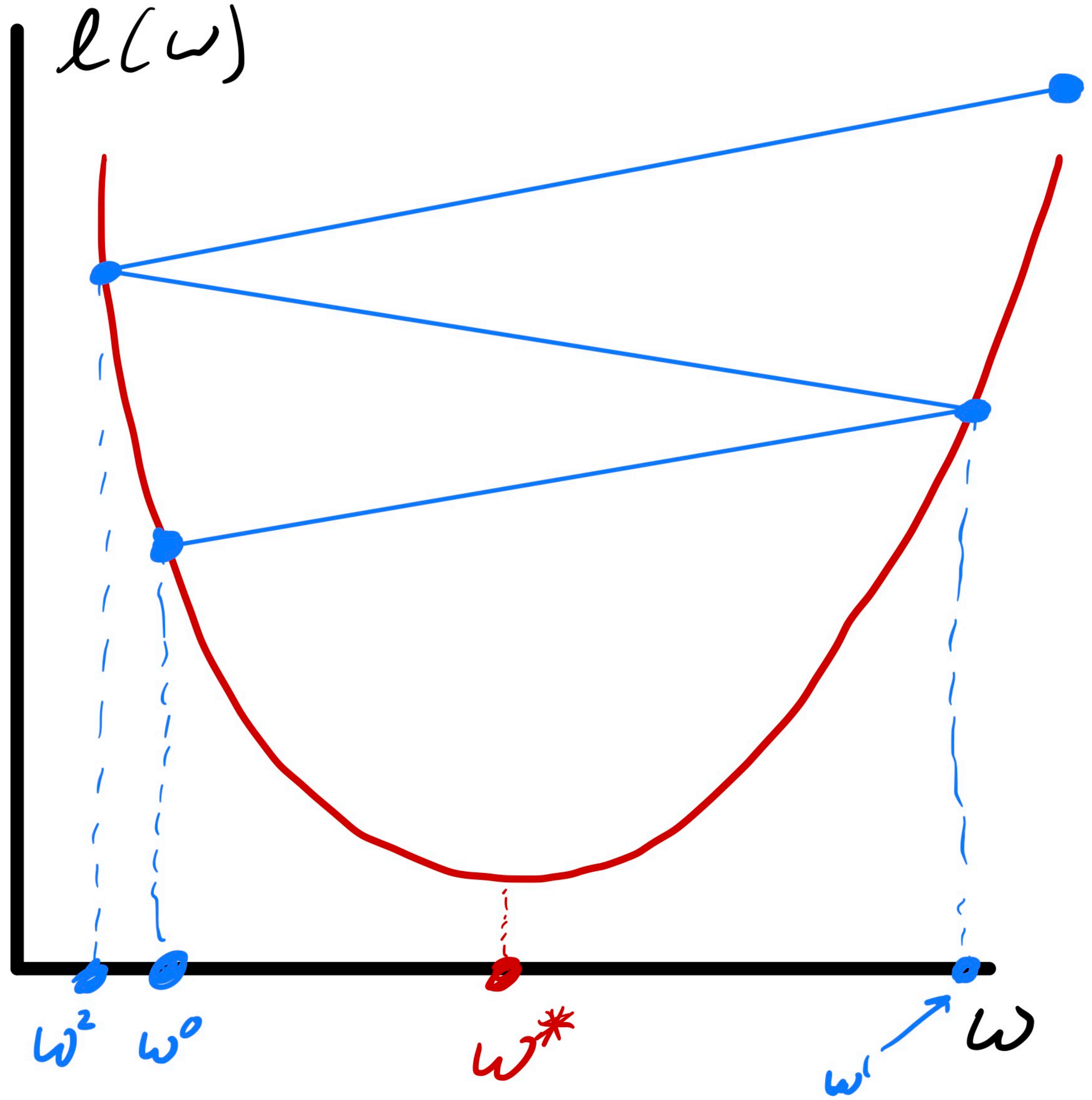
$$\ell(w - \eta \nabla \ell(w)) = \ell(w) - \eta \nabla \ell(w)^\top (\nabla \ell(w))$$

$$\|\nabla \ell(w)\|_2^2 > 0$$

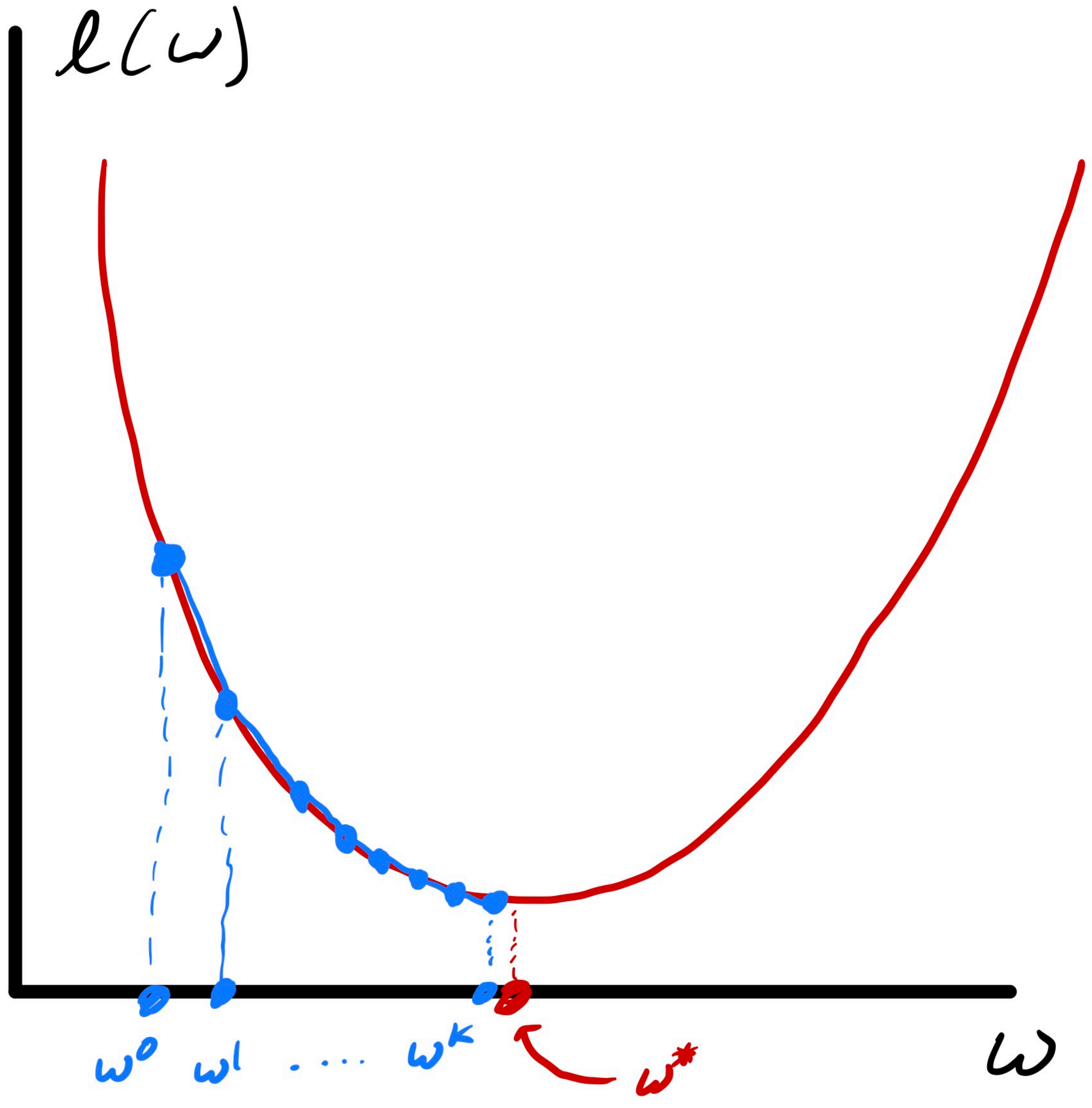
i.e., w/ sufficiently small η, GD decrease obj value if $\nabla \ell(w) \neq 0$!

How to set learning rate η in practice?

Large η typically is bad and can lead to diverge



In theory, for convex loss, $\eta = c/\sqrt{k}$ guarantees convergence



Let's summarize by applying GD to logistic regression

Recall the objective for LR:

$$\min_{w} \sum_{i=1}^{n} \ln\left[1 + \exp\left(-y_i(w^\top x_i)\right)\right] + \lambda\|w\|_2^2$$

Initialize $w^0 \in \mathbb{R}^d$

Iterate until convergence:

1. Compute gradient $g^t = \sum_i \frac{\exp(-y_i x_i^\top w^t)(-y_i x_i)}{1 + \exp(-y_i x_i^\top w^t)} + 2\lambda w^t$
2. Update (GD): $w^{t+1} = w^t - \eta g^t$