Boosting

Announcements

Recap on Bagging

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$$\hat{h} = \sum_{i=1}^{m} \hat{h}_i / m$$

Outline of Today

1. Gradient Descent without accurate gradient

2. Boosting as Approximate Gradient Descent

3. Example: the AdaBoost Algorithm

Gradient Descent without an accurate gradient

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When η is small and $g_t \neq 0$, we know $L(y_{t+1}) < L(y_t)$

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Approximate Gradient descent:

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Q: Under what condition of $\hat{g}_t$, can we still guarantee $L(y_{t+1}) < L(y_t)$?

A: As long as $\left\langle \hat{g}_t, \nabla L(y_t) \right\rangle > 0$

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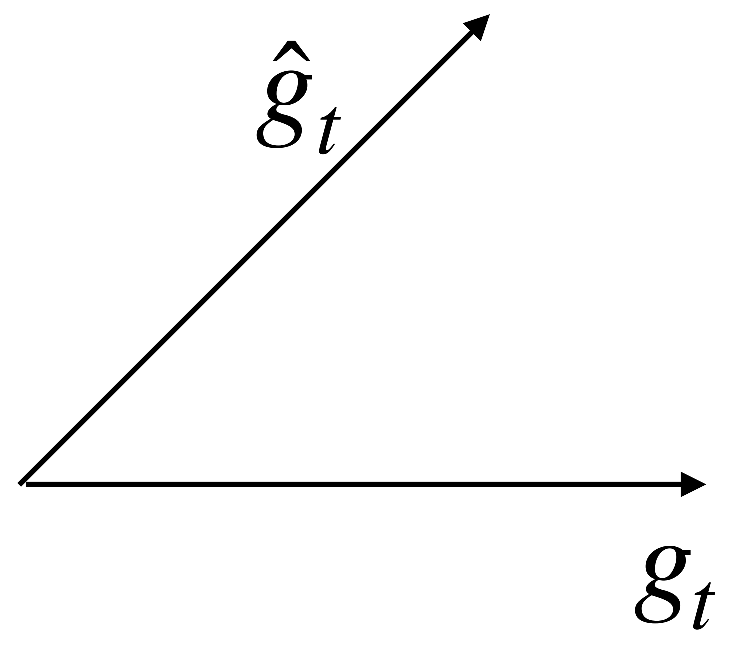

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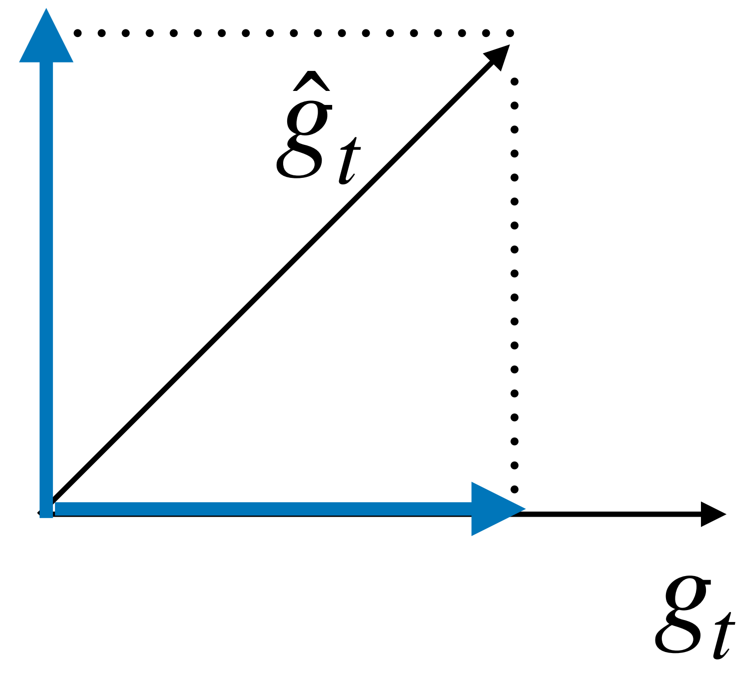

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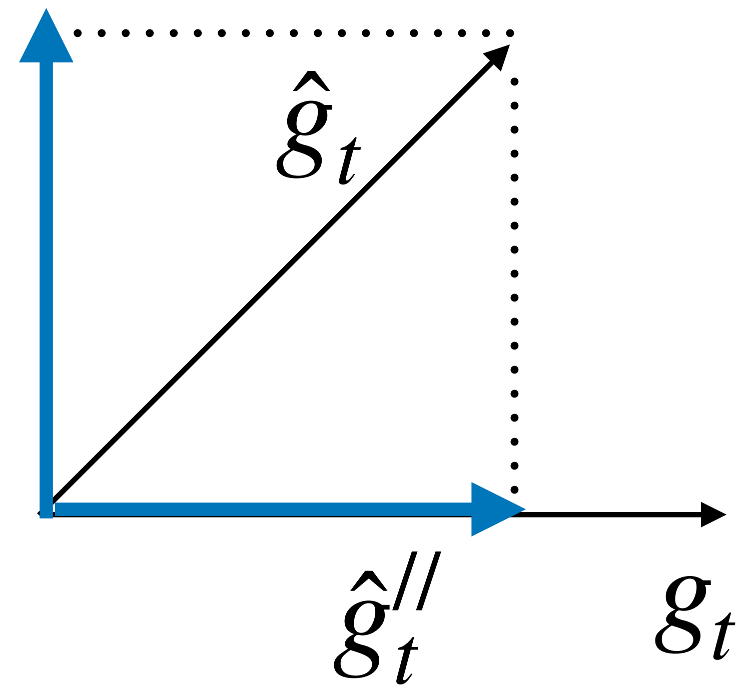

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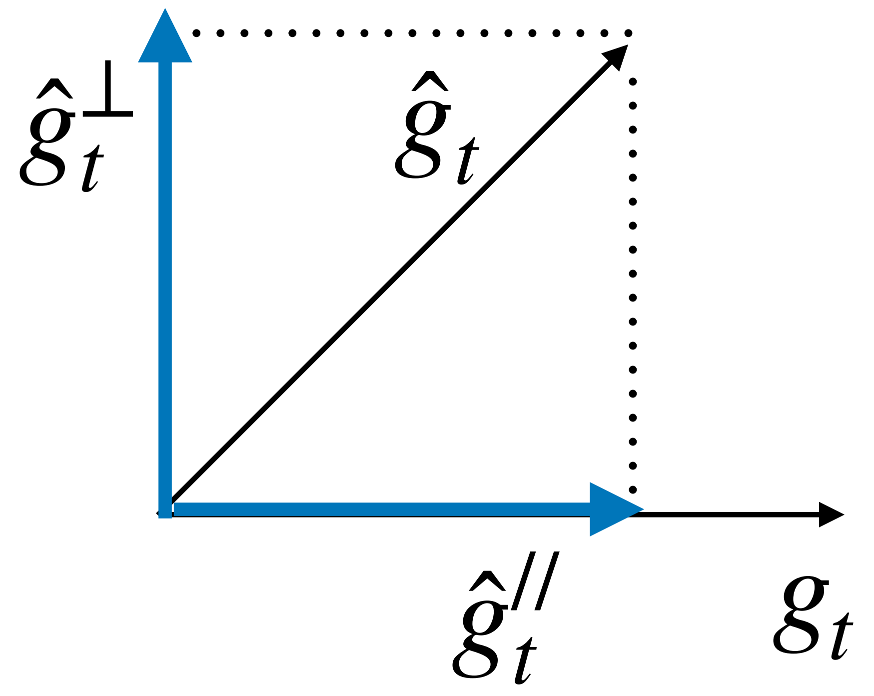

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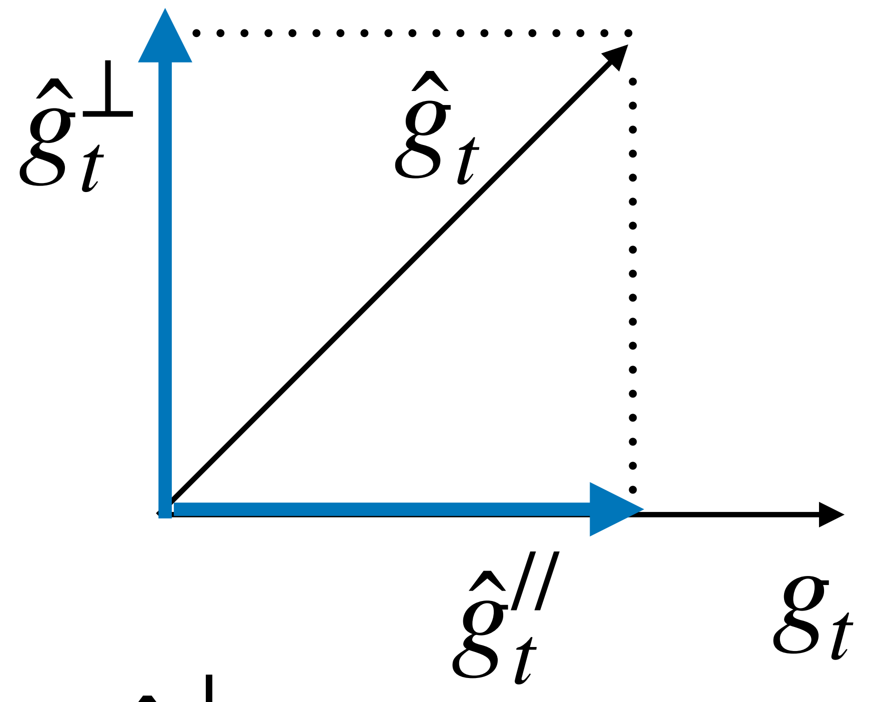

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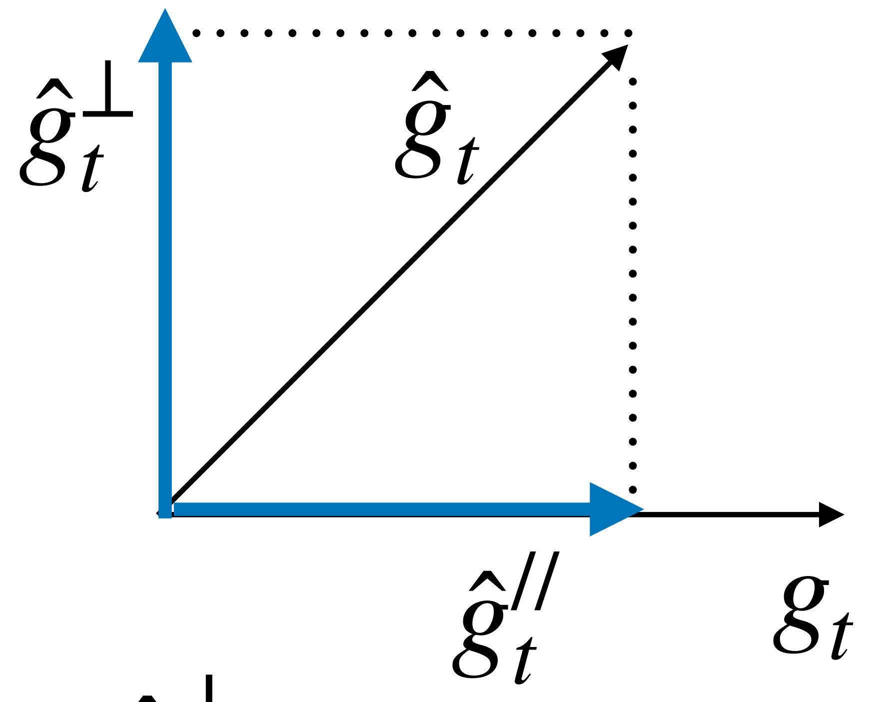

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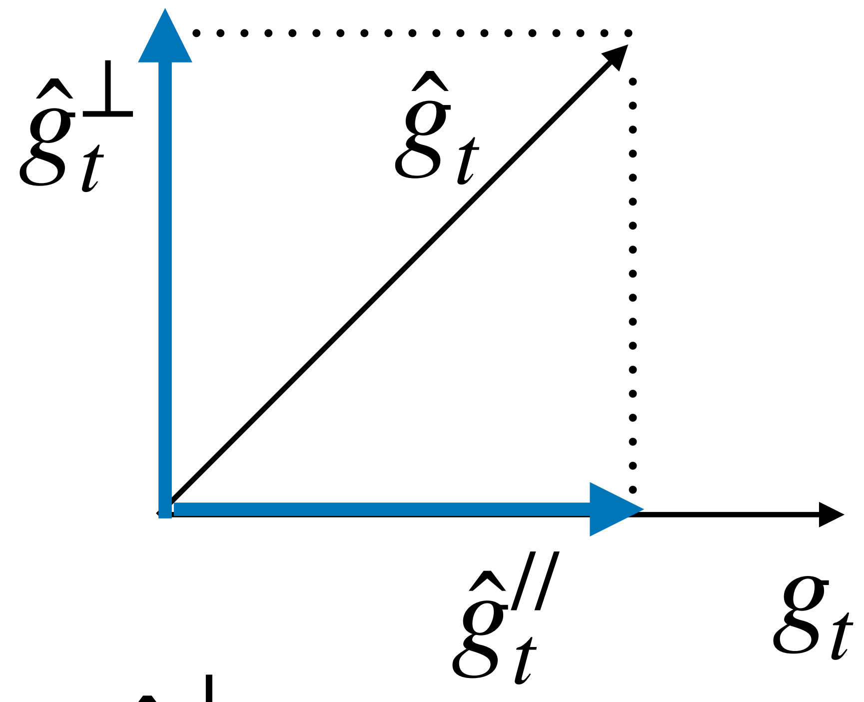
Prove this via first order Taylor expansion and the fact that $\hat{g}_t^\top g_t > 0$



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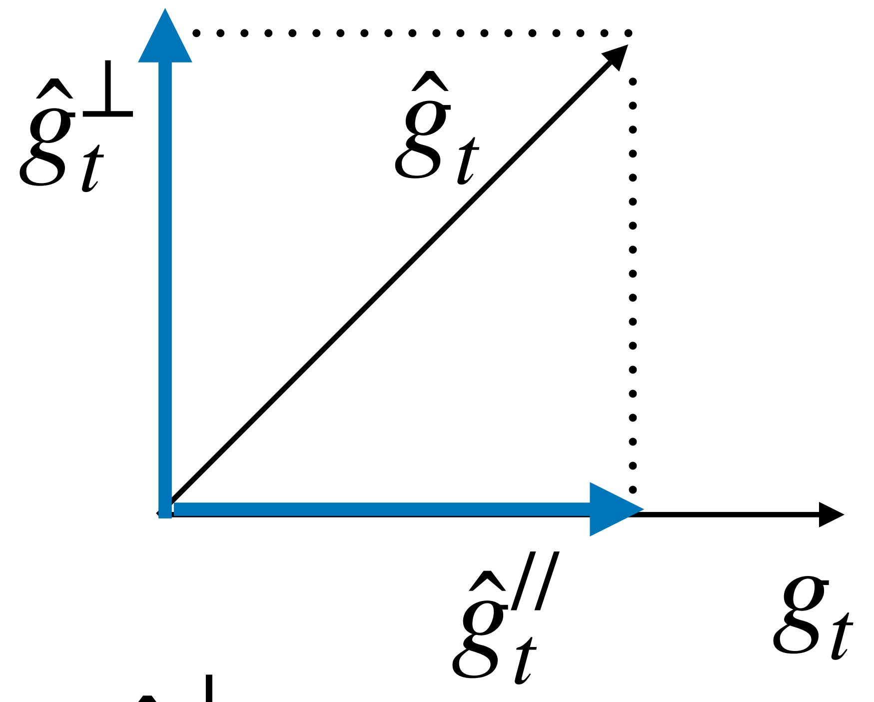

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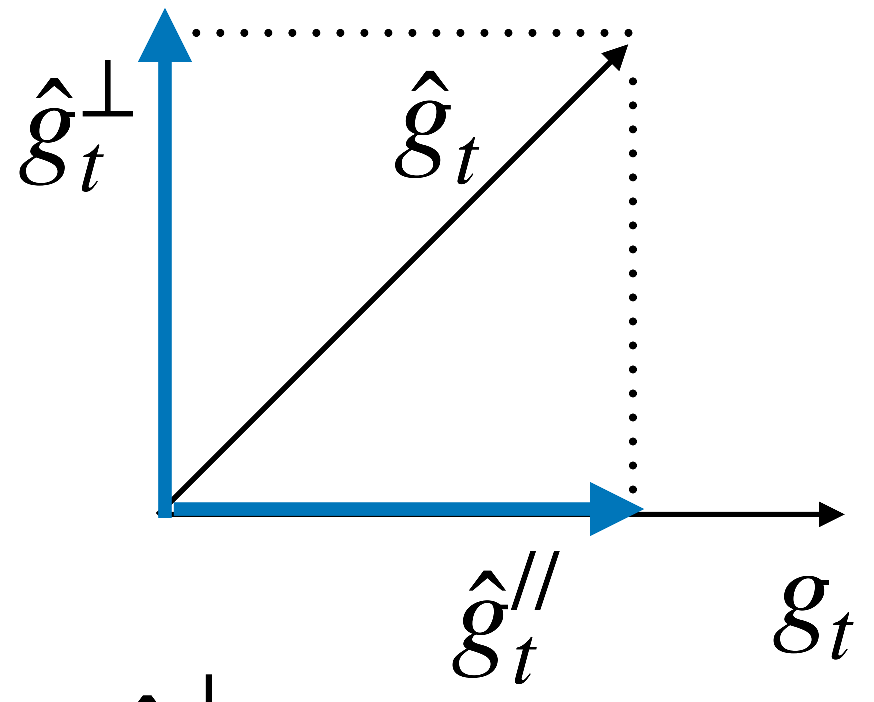

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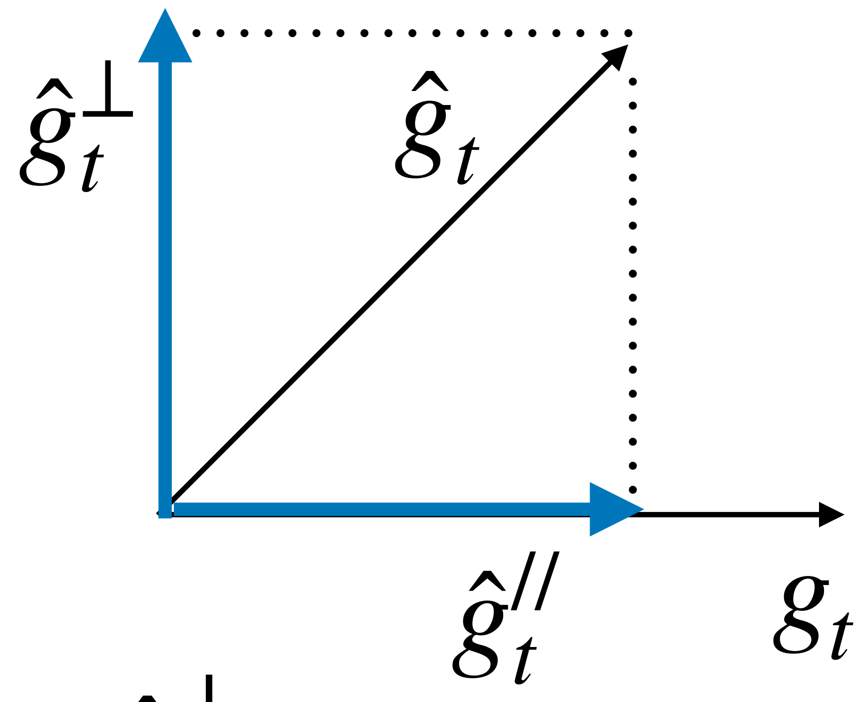

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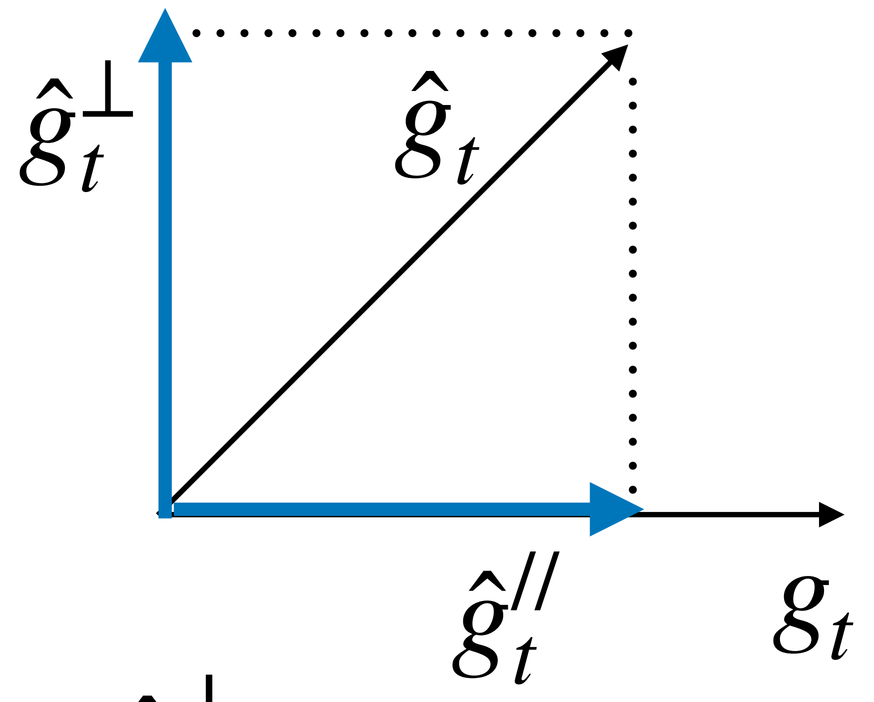
Positive since $\alpha > 0$



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Key question that Boosting answers:

Can weak learners be combined together to generate a strong learner with low bias?

(Weak learners: classifiers whose accuracy is slightly above 50%)

Setup

We have a binary classification data $\mathscr{D} = \{x_i, y_i\}_{i=1}^{n}, (x_i, y_i) \sim P$

Hypothesis class $\mathscr{H},$ hypothesis $h : X \mapsto \{-1, +1\}$

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Goal: learn an ensemble $H(x) = \sum_{t=1}^{T} \alpha_t h_t(x)$, where $h_t \in \mathscr{H}$

The Boosting Algorithm

Initialize $H_1 = h_1 \in \mathscr{H}$

For t = 1 …

Find a new classifier h_{t+1}, s.t., $H_{t+1} = H_t + \alpha h_{t+1}$ has smaller training error

Training weak learners

Denote $\hat{\mathbf{y}} = \left[H_t(x_1), H_t(x_2), \ldots, H_t(x_n)\right]^\top \in \mathbb{R}^n$

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A: no, we want find $\hat{\mathbf{y}}$ that minimizes L, but it needs to be from some ensemble H

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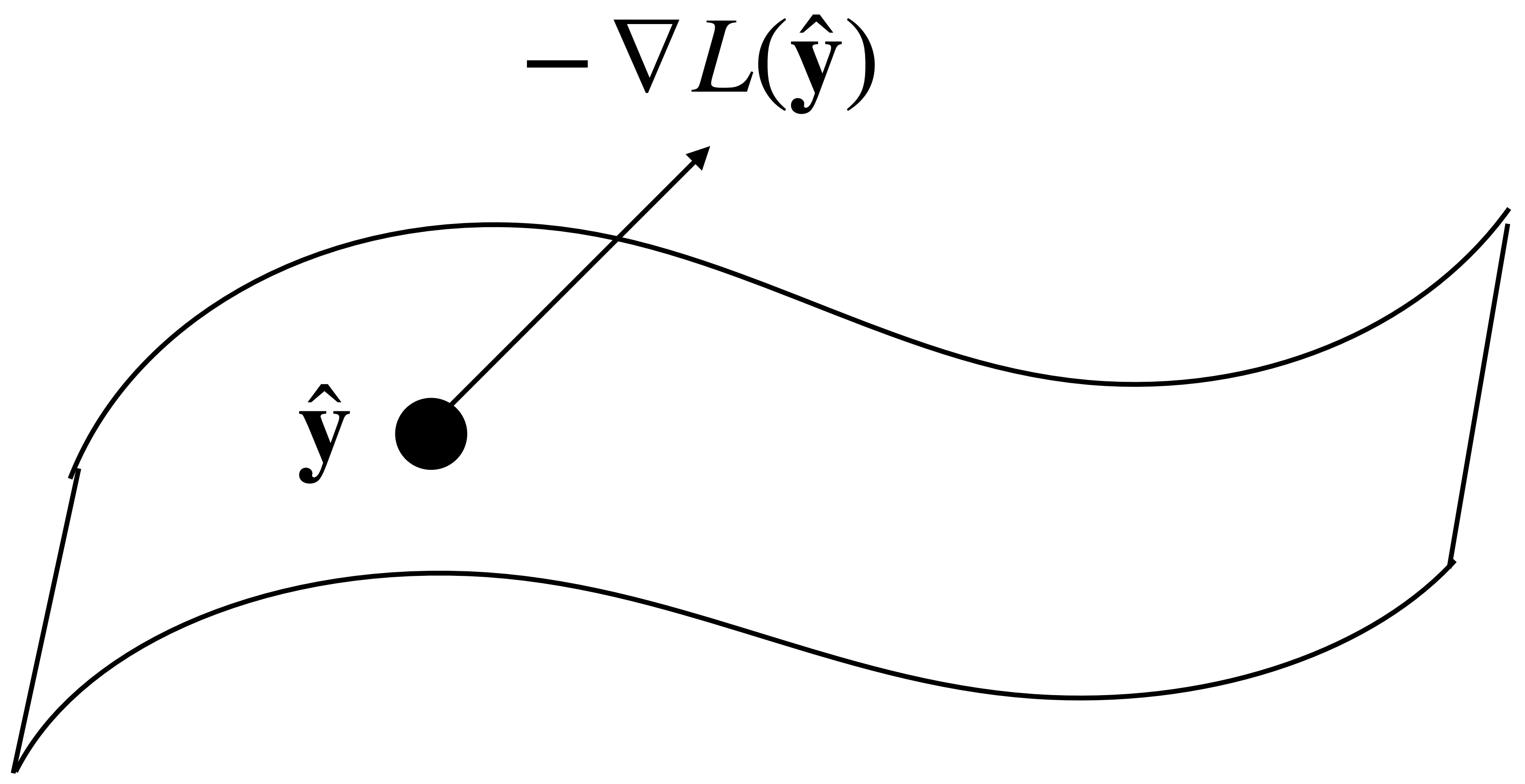
Let us compute $\nabla L(\hat{\mathbf{y}}) \in \mathbb{R}^n$ — the ideal descent direction



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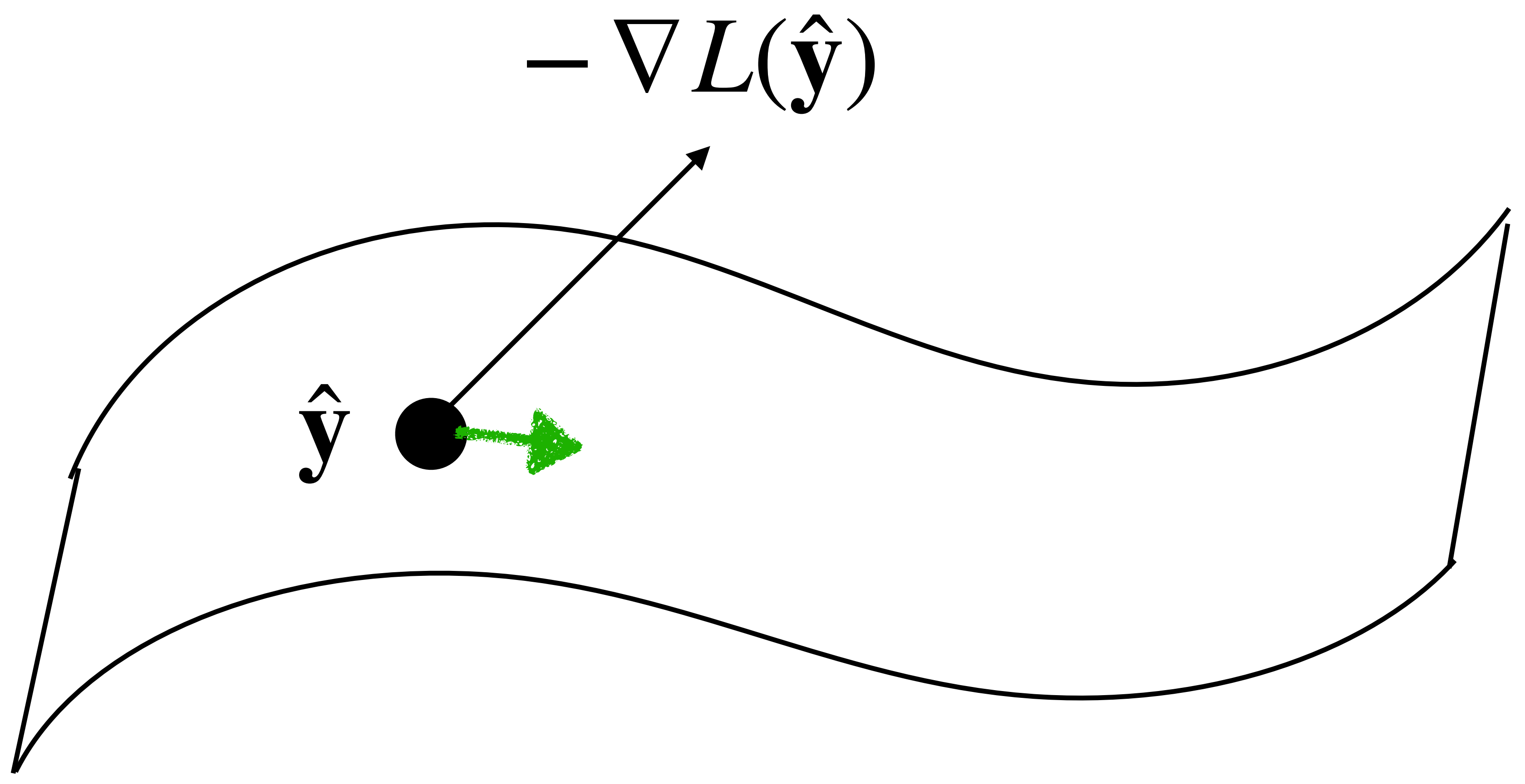

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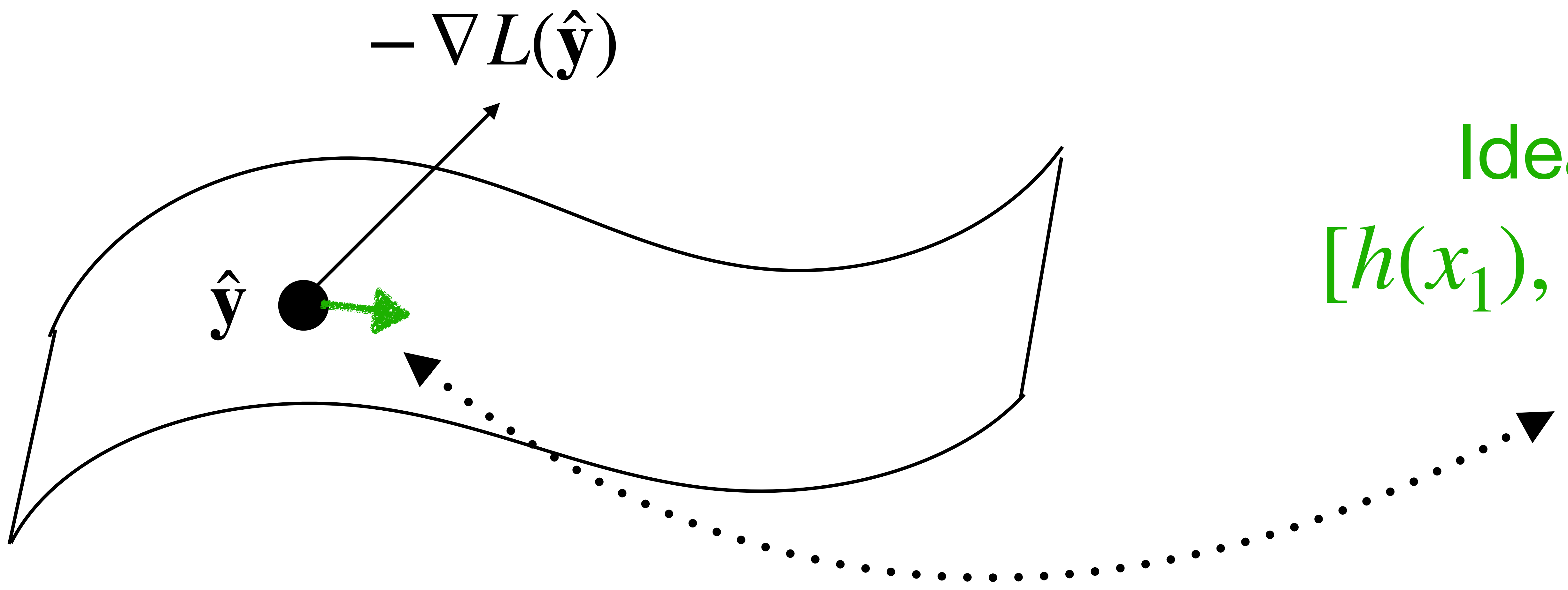
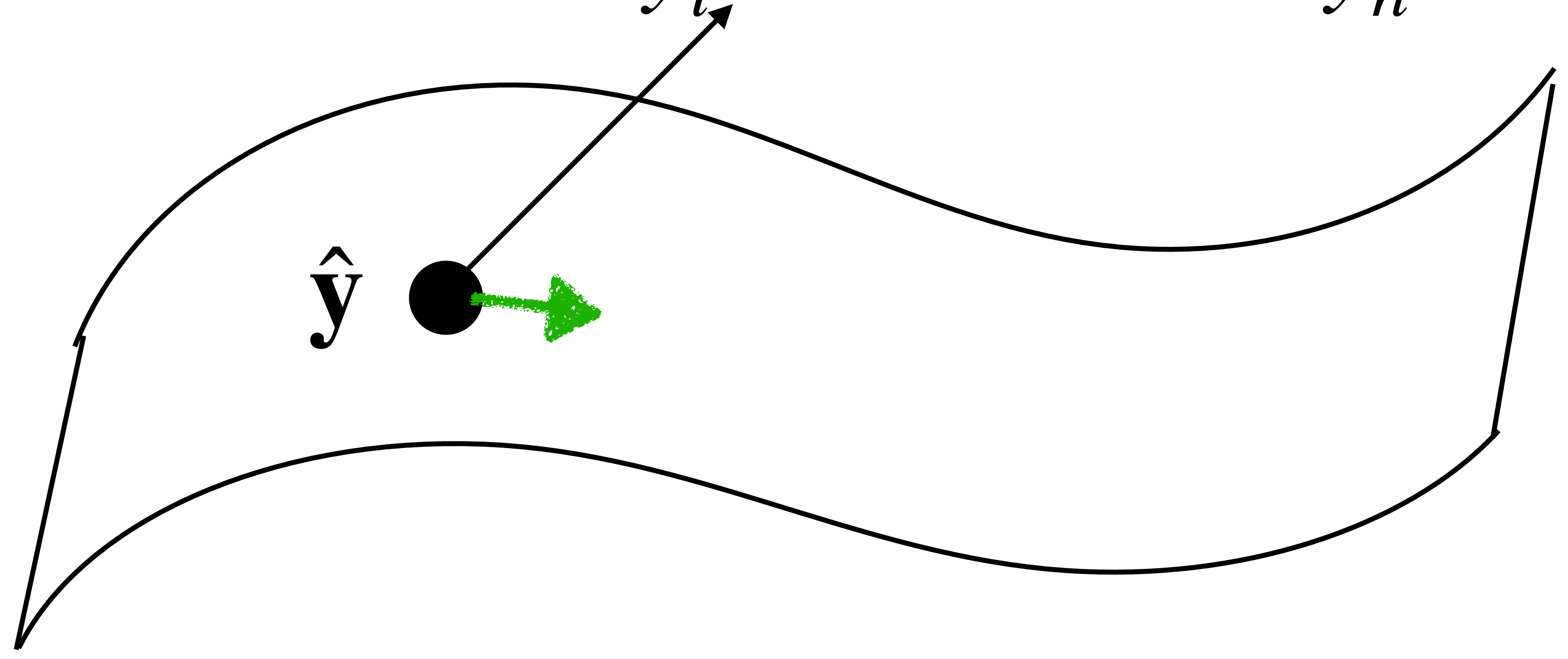


Idea: find a $h \in \mathscr{H}$, such that $[h(x_1), \ldots h(x_n)]^\top$ is close to $-\nabla L(\hat{\mathbf{y}})$

Training weak learners

$$-\nabla L(\hat{\mathbf{y}}) = [-\frac{\partial \ell(\hat{y}_i, y_i)}{\partial \hat{y}_i}, \ldots, -\frac{\partial \ell(\hat{y}_n, y_n)}{\partial \hat{y}_n}]^\top$$



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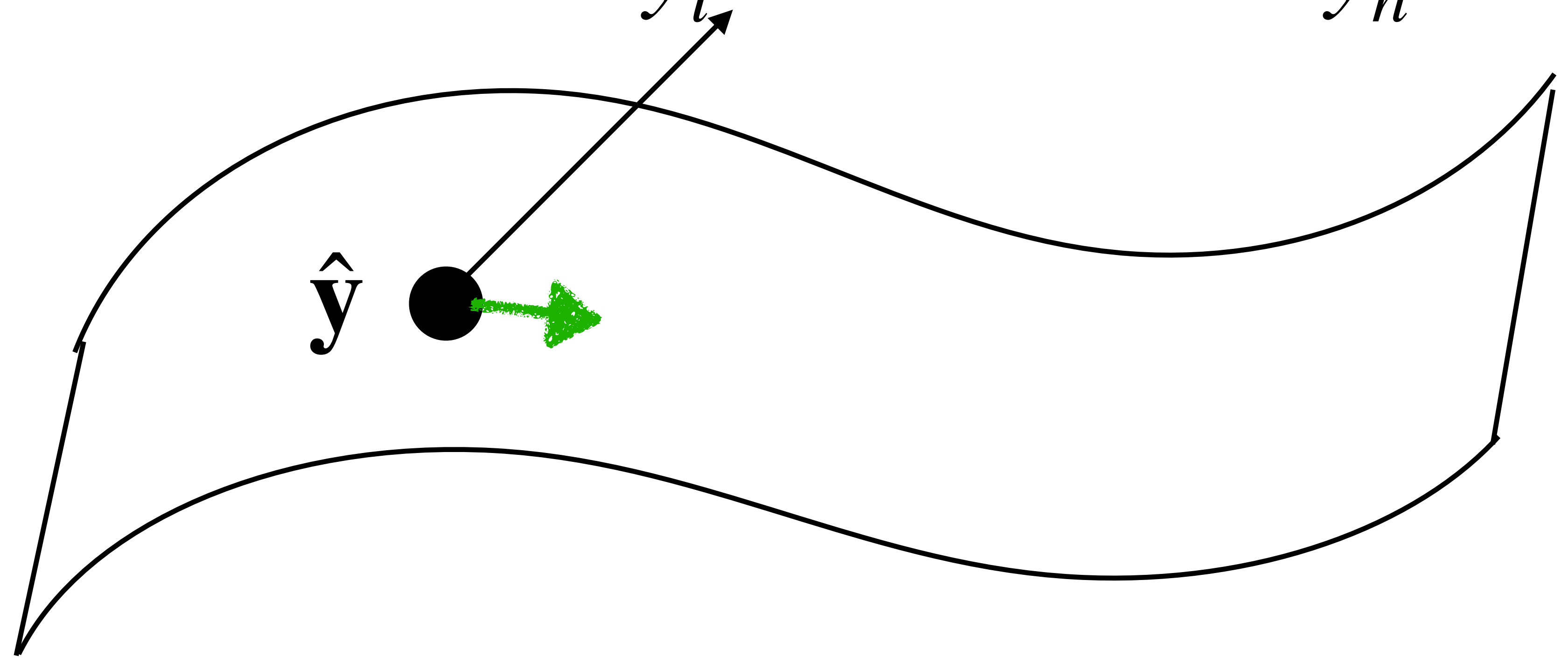

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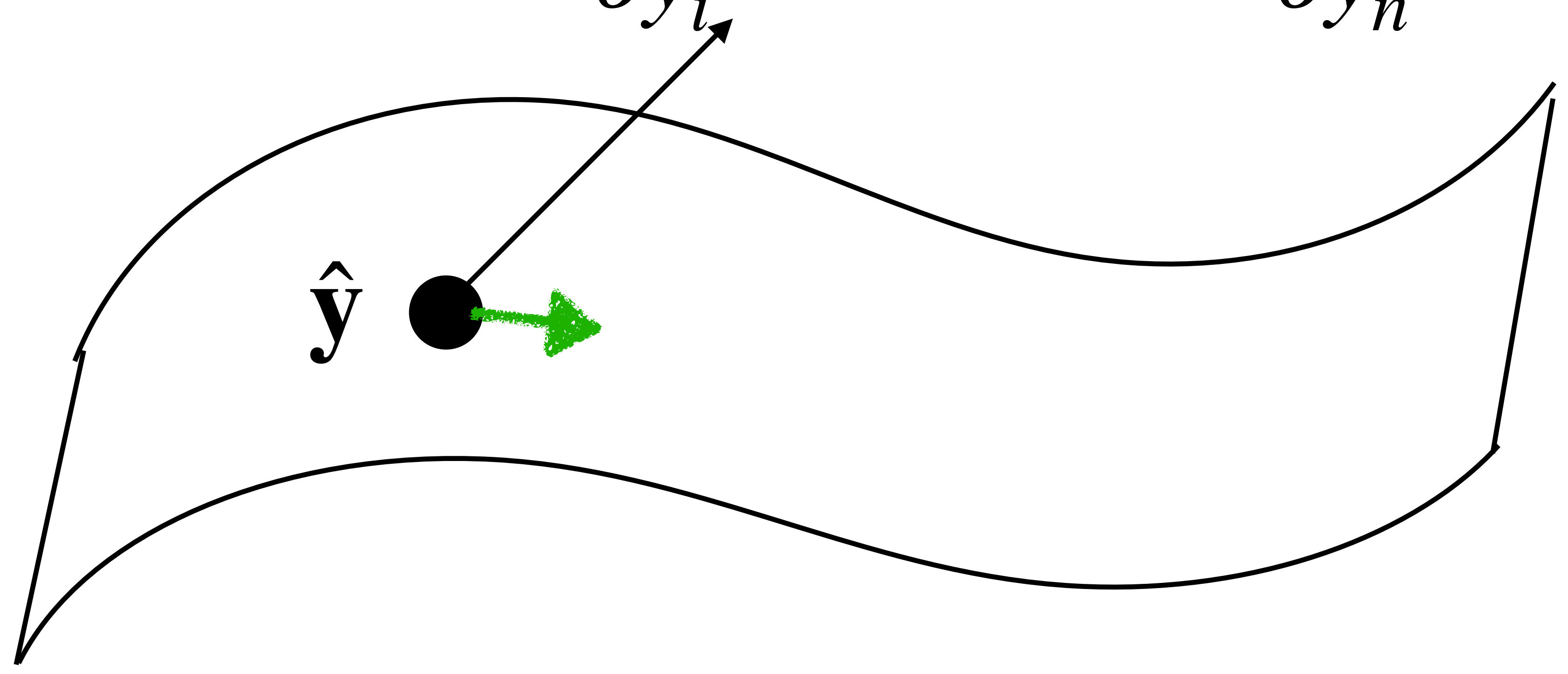

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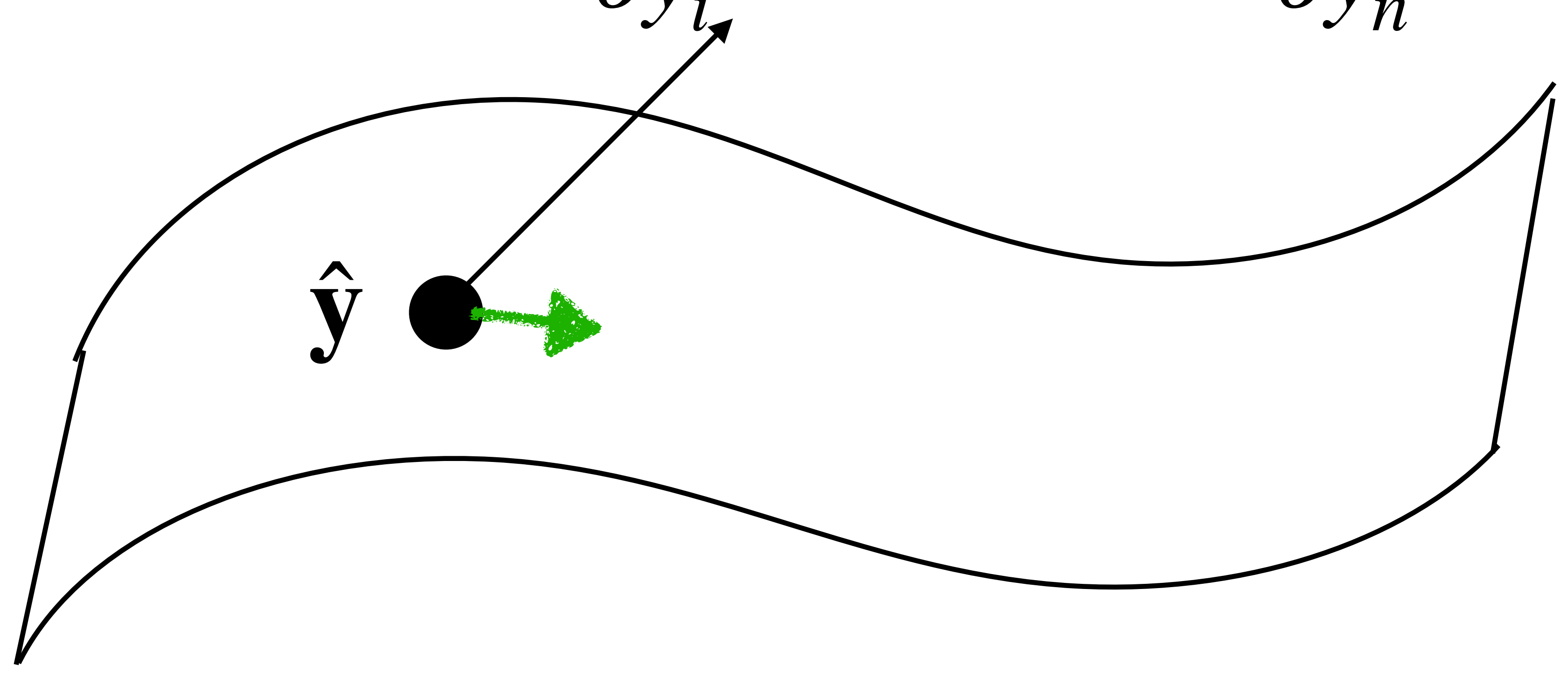

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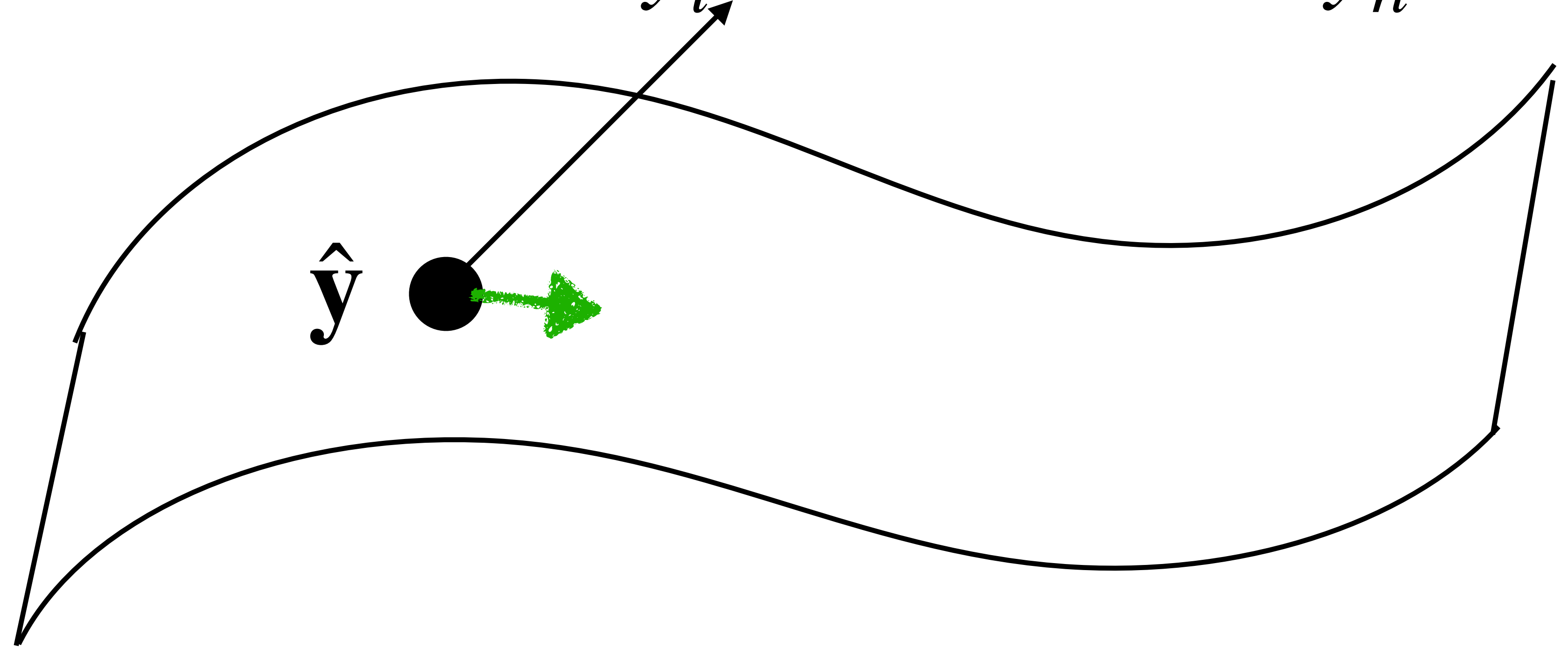

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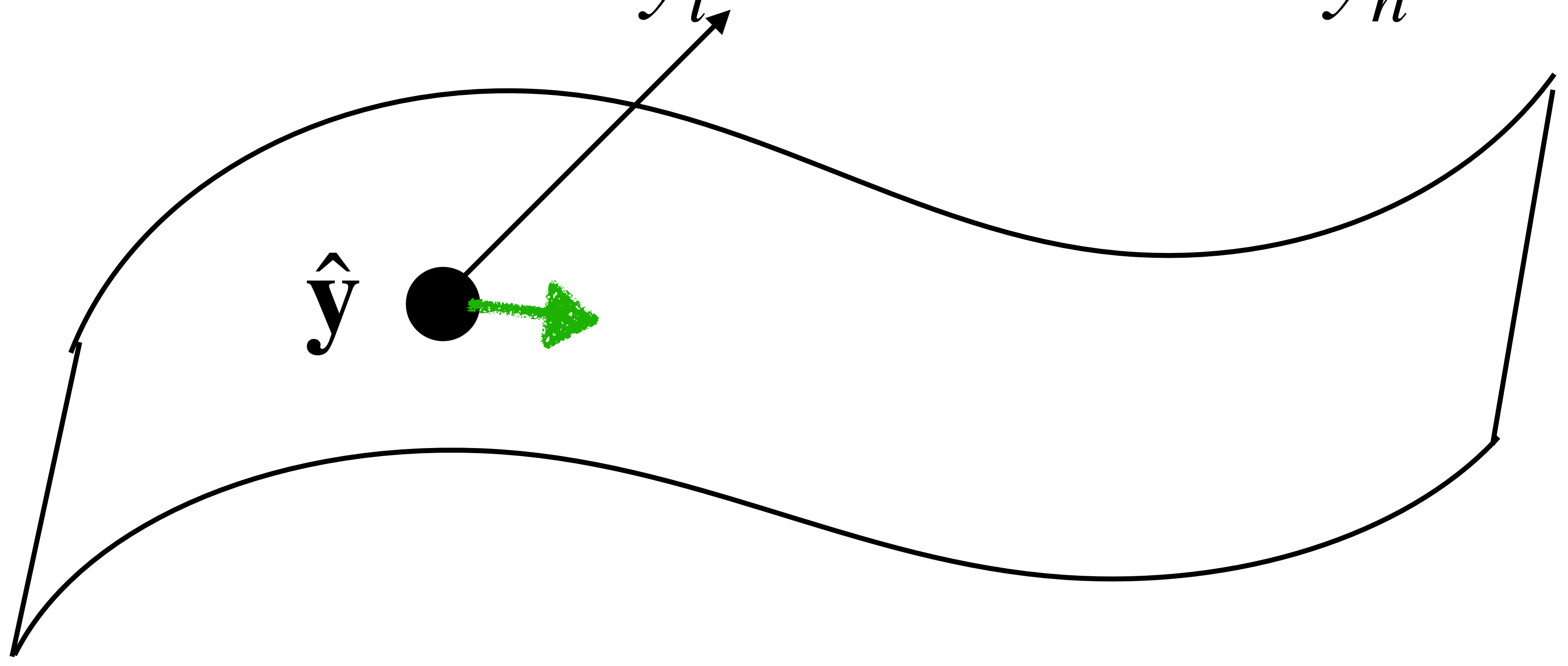

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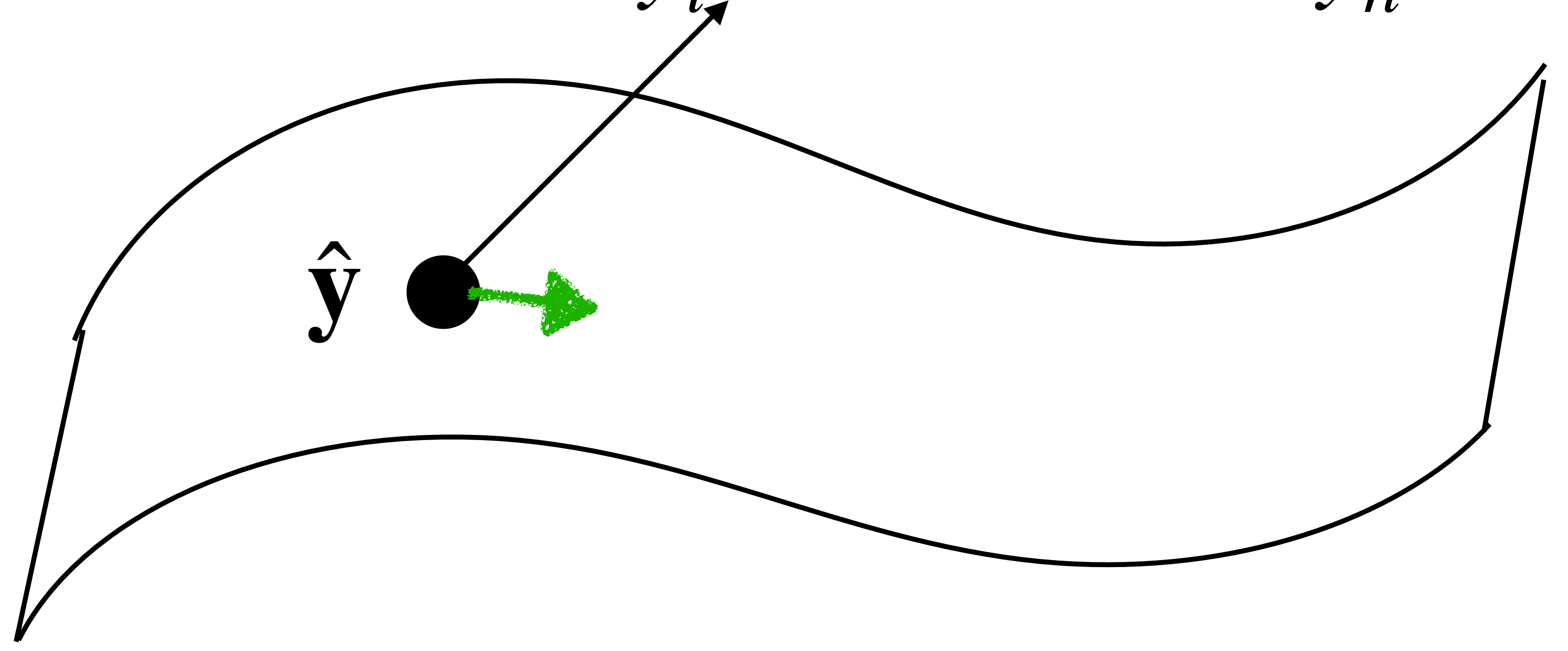

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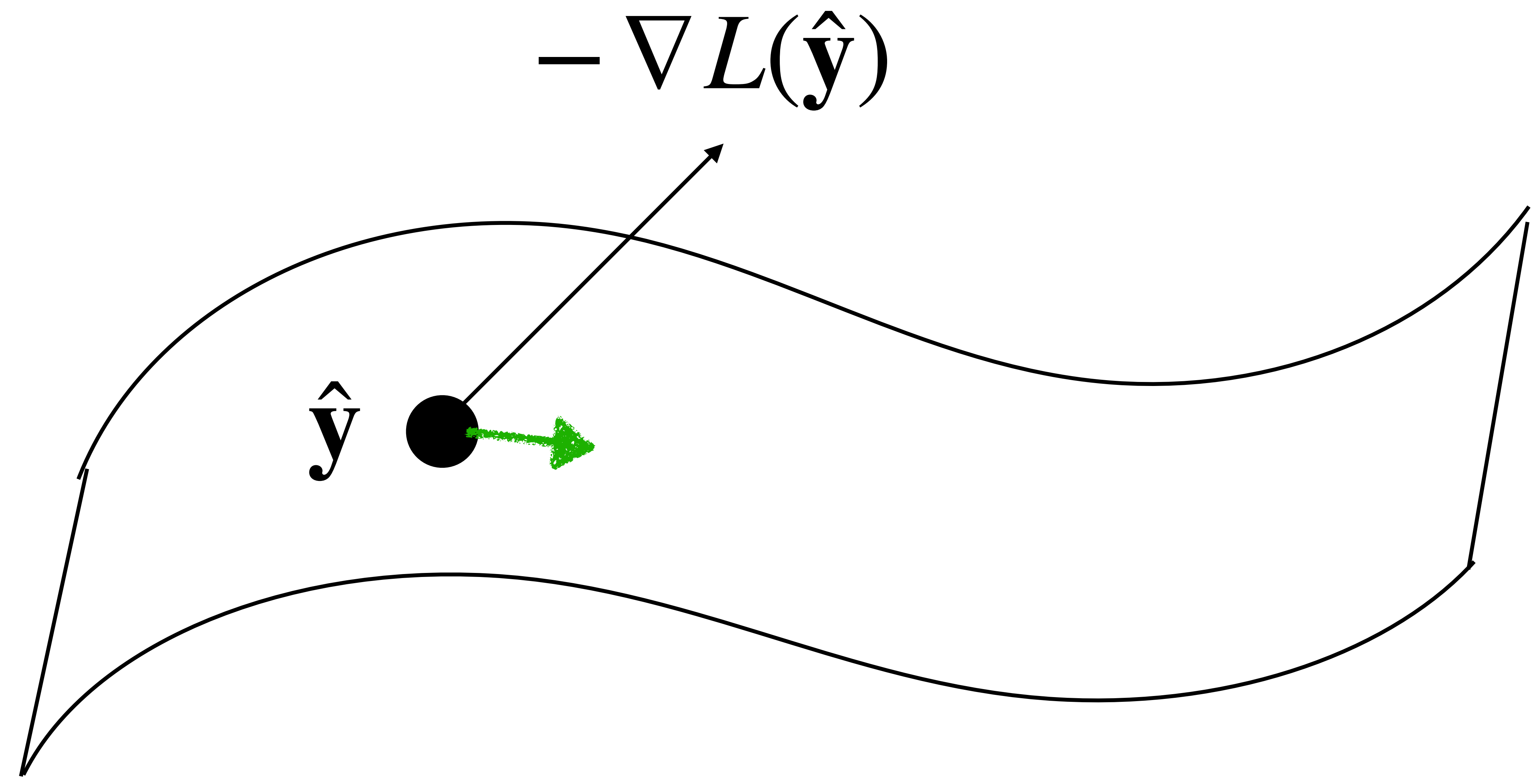
$$= \arg\min_{h \in \mathcal{H}} \sum_{i=1}^{n} |w_i| \cdot \mathbf{1}(h(x_i) = \mathrm{sign}(w_i)) \quad = \arg\min_{h \in \mathcal{H}} \sum_{i=1}^{n} |w_i| \cdot \mathbf{1}(h(x_i) \neq -\mathrm{sign}(w_i))$$

Turned it to a weighted classification problem!

Training weak learners

Finding $[h(x_1), \ldots, h(x_n)]^\top$ that is close to $-\nabla L(\hat{\mathbf{y}})$ can be done via weighted binary classification:



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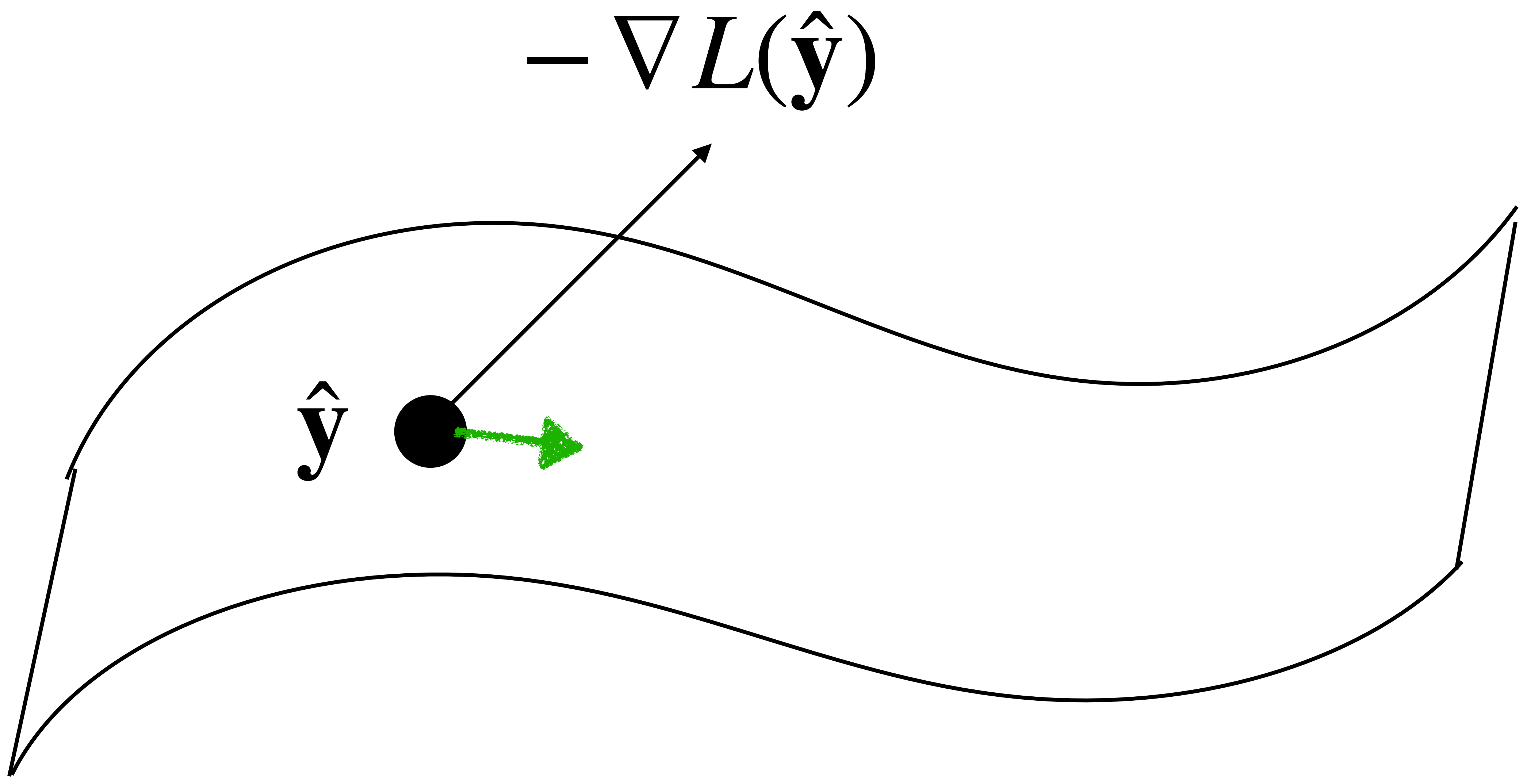

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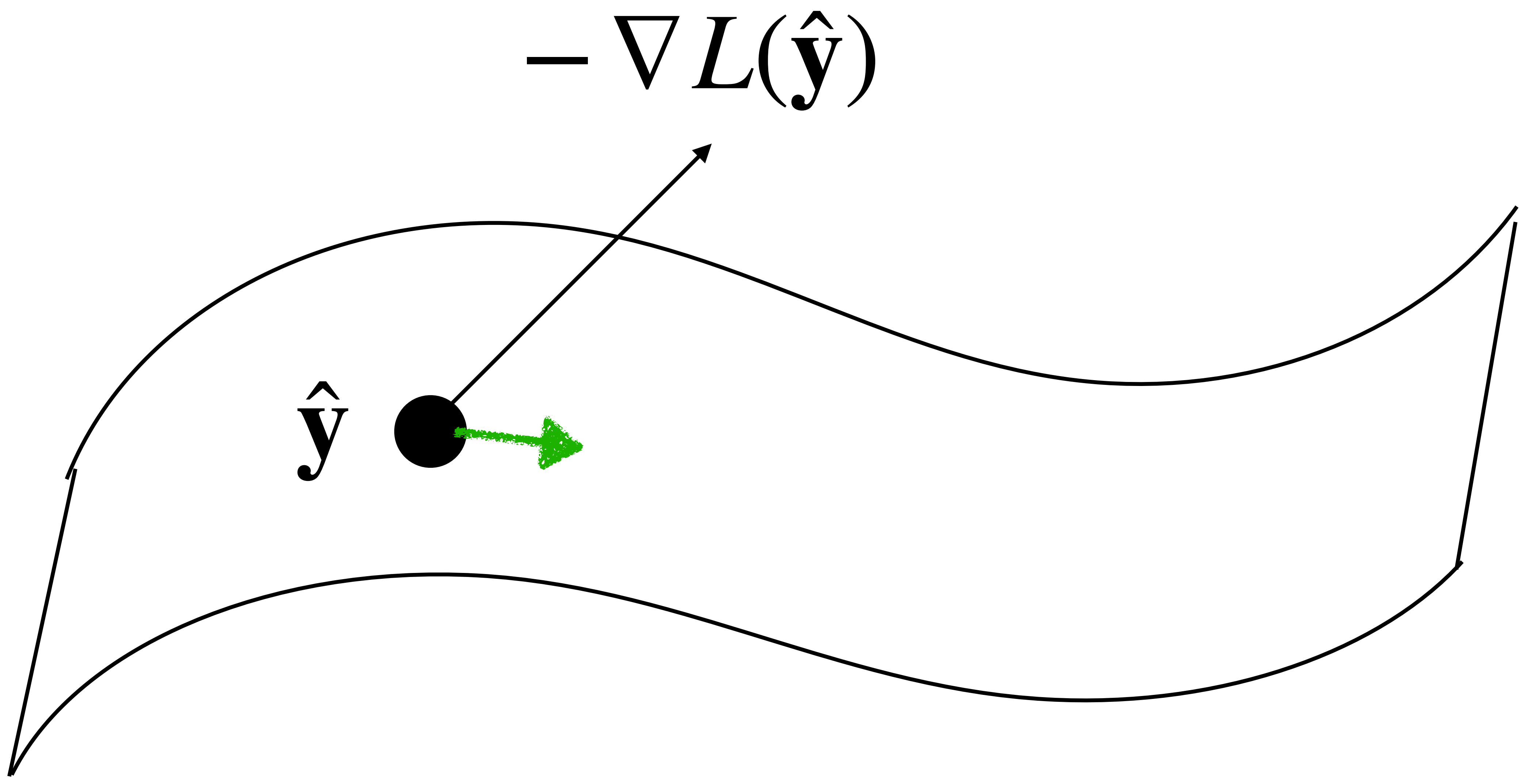

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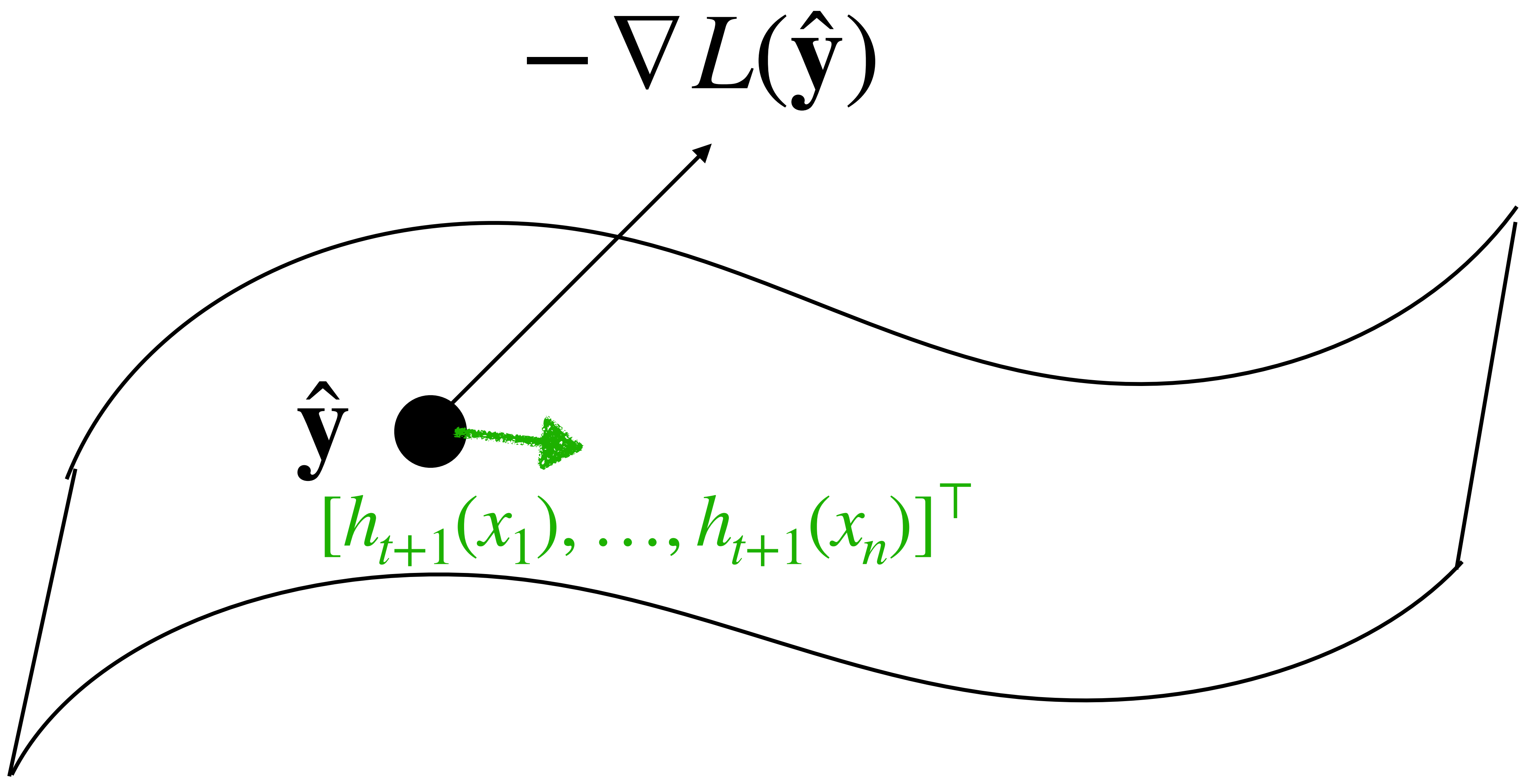

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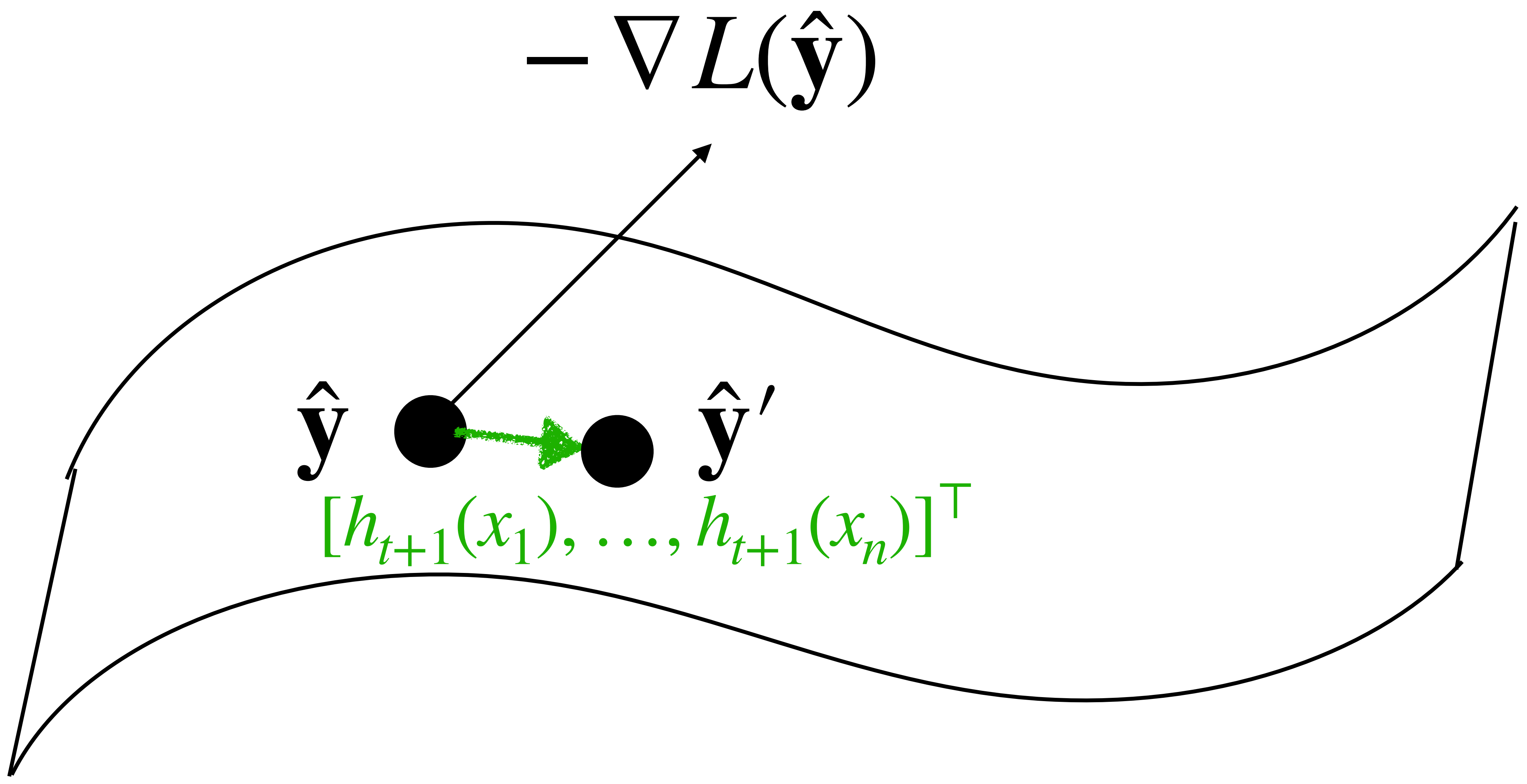

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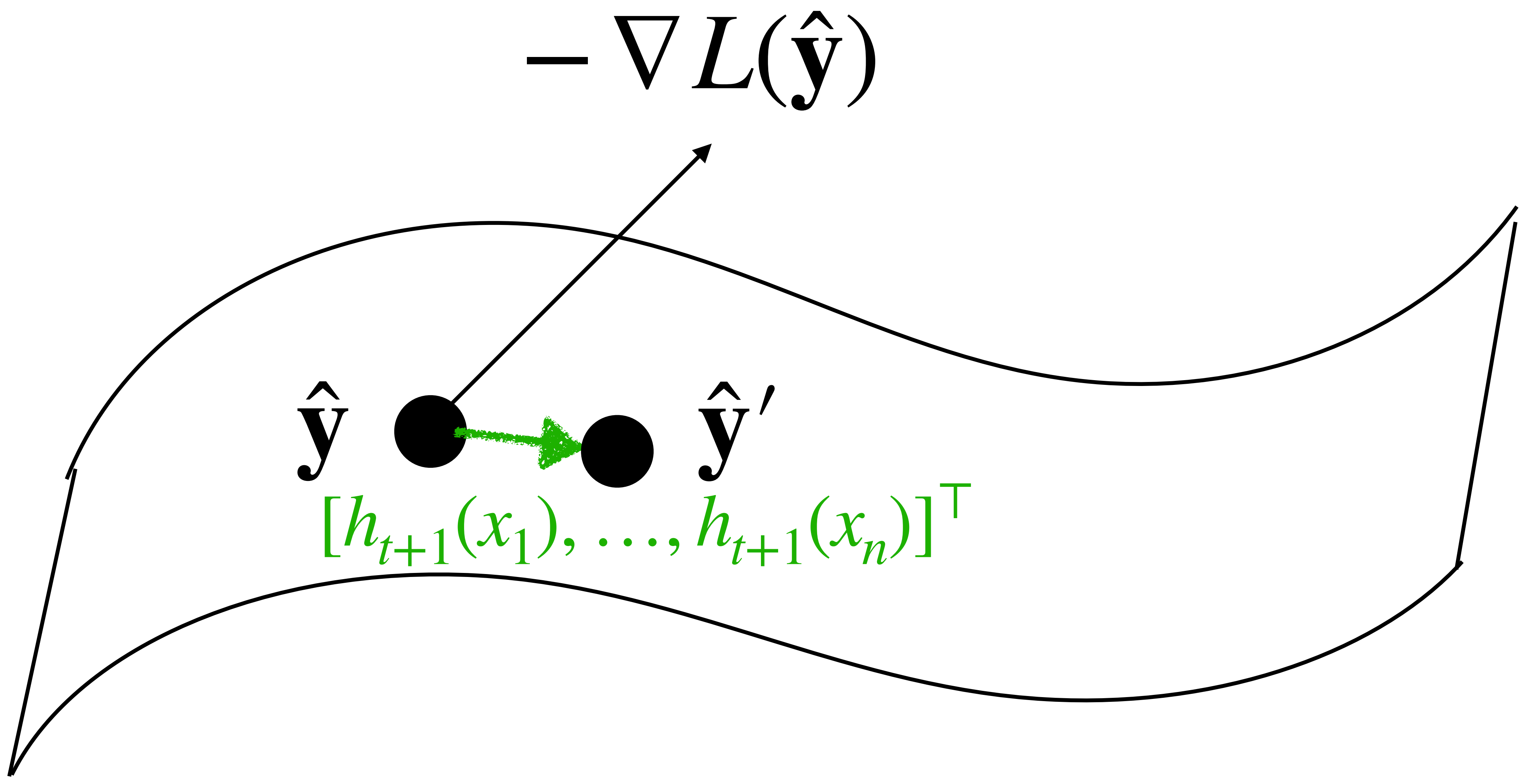

$$\hat{\mathbf{y}}' = \hat{\mathbf{y}} + \alpha[h_{t+1}(x_1), \ldots, h_{t+1}(x_n)]^\top$$

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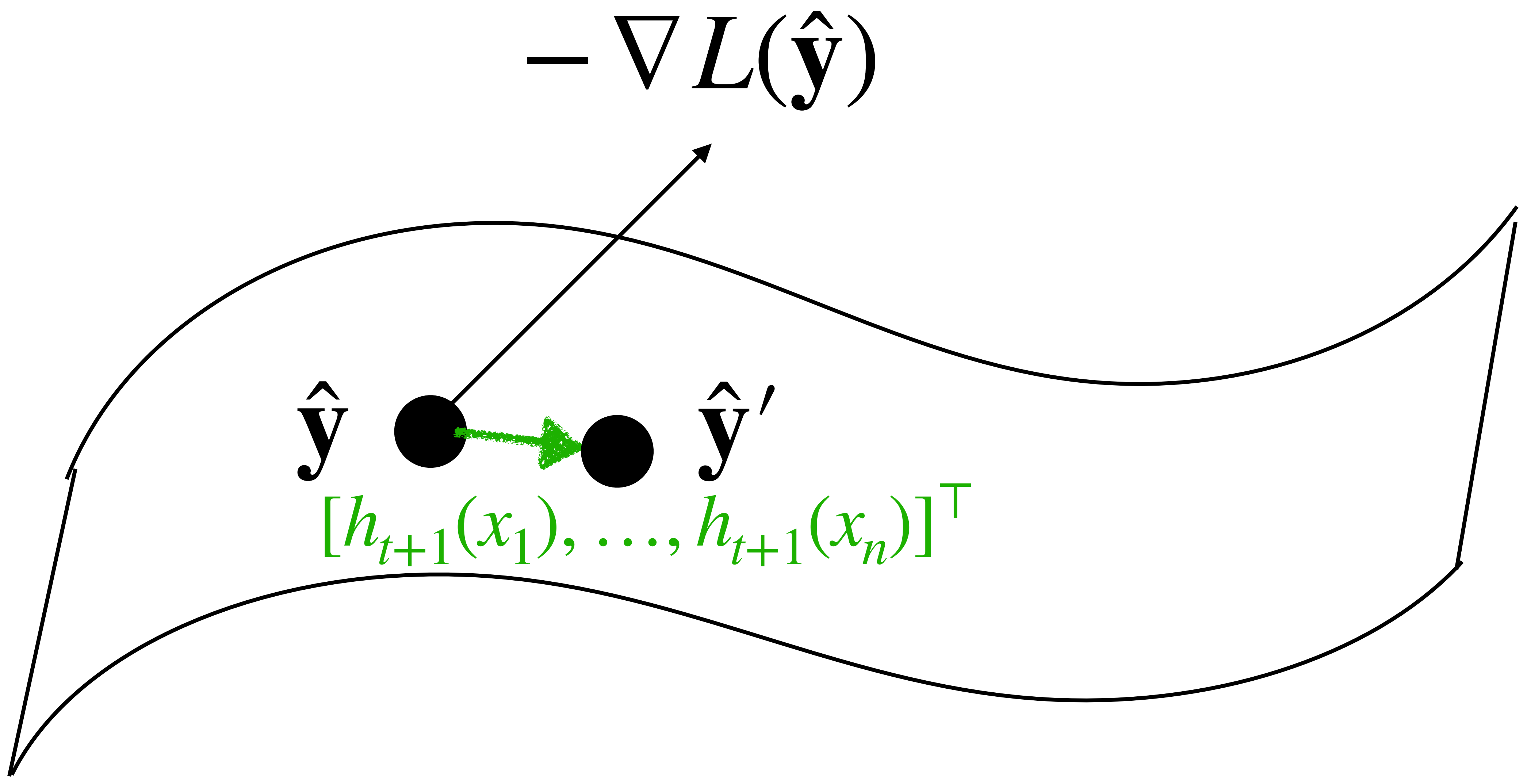

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The Boosting Algorithm Revisit

Initialize $H_1 = h_1 \in \mathscr{H}$

For t = 1 …

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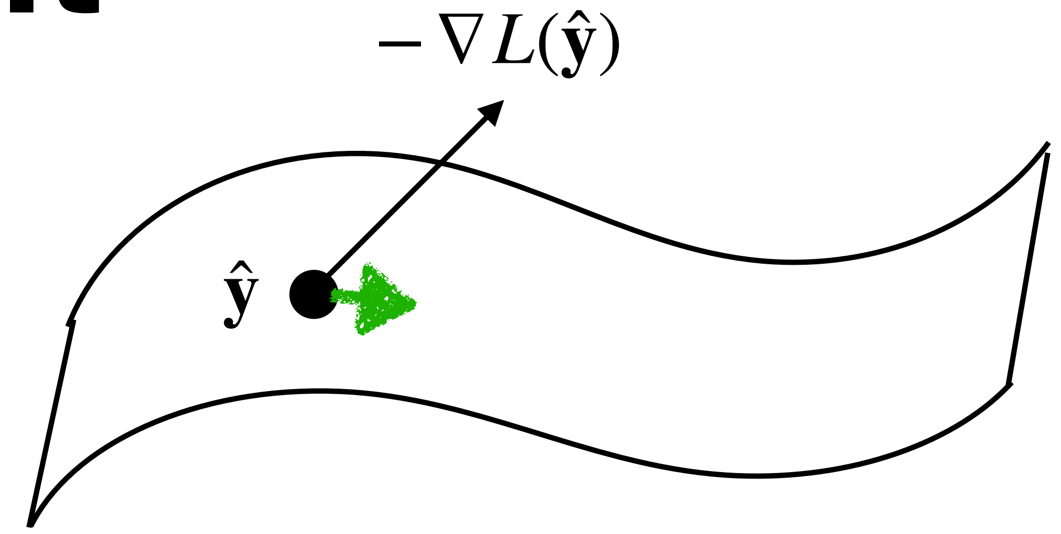

$$\arg\max_{h \in \mathcal{H}} (-\nabla L(\hat{\mathbf{y}}))^\top \begin{bmatrix} h(x_1) \\ h(x_2) \\ \cdots \\ h(x_n) \end{bmatrix}$$

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Outline of Today

1. Gradient Descent without accurate gradient

2. Boosting as Approximate Gradient Descent

3. Example: the AdaBoost Algorithm

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Binary classification on weighted data

$\widetilde{\mathcal{D}} = \{p_i, x_i, y_i\}$, where $\sum_i p_i = 1, p_i \geq 0, \forall i$

Q: what does it mean if p_i is large?

Compute learning rate

Select the best learning rate α

$$h_{t+1} = \arg\min_{h \in \mathcal{H}} \sum_{i=1}^{n} p_i \cdot \mathbf{1}(h(x_i) \neq y_i) \qquad H_{t+1} = H_t + \alpha h_{t+1}$$

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Find the best learning rate via optimization:

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Compute the derivative wrt α, set it to zero, and solve for α

Put everything together: AdaBoost

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For t = 1 …

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// the best $\alpha = 0.5 \ln((1-\epsilon)/\epsilon)$

Analysis of AdaBoost

From weak learners to a strong learner that minimizes training error

The definition of Weak learning

Each weaker learning optimizes its own data:

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Q: assume $\mathscr{H}$ is symmetric, i.e., $h \in \mathscr{H}$ iff $-h \in \mathscr{H}$, why does the above always hold?

Weaker learnability implies approximating gradient well

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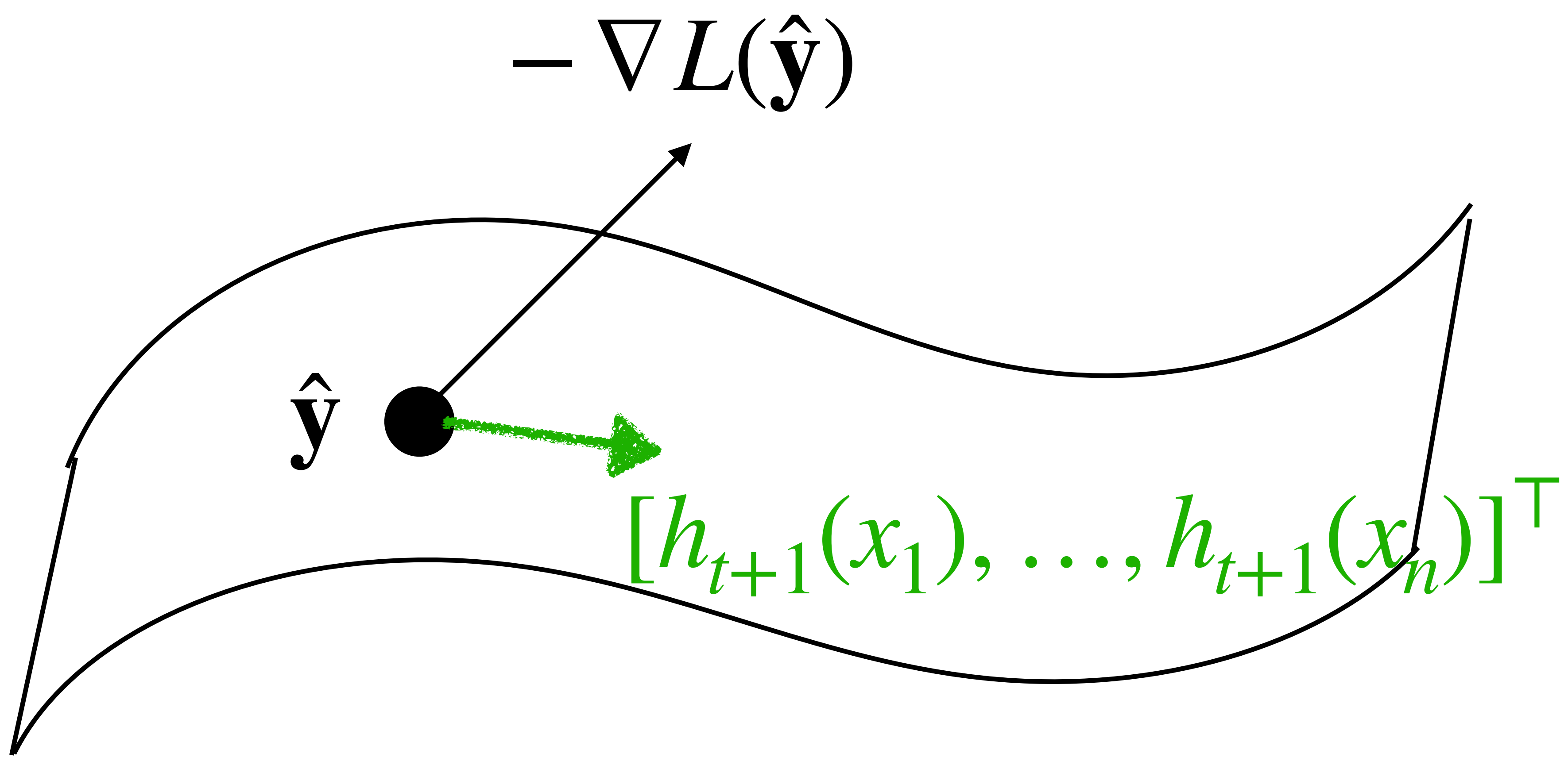

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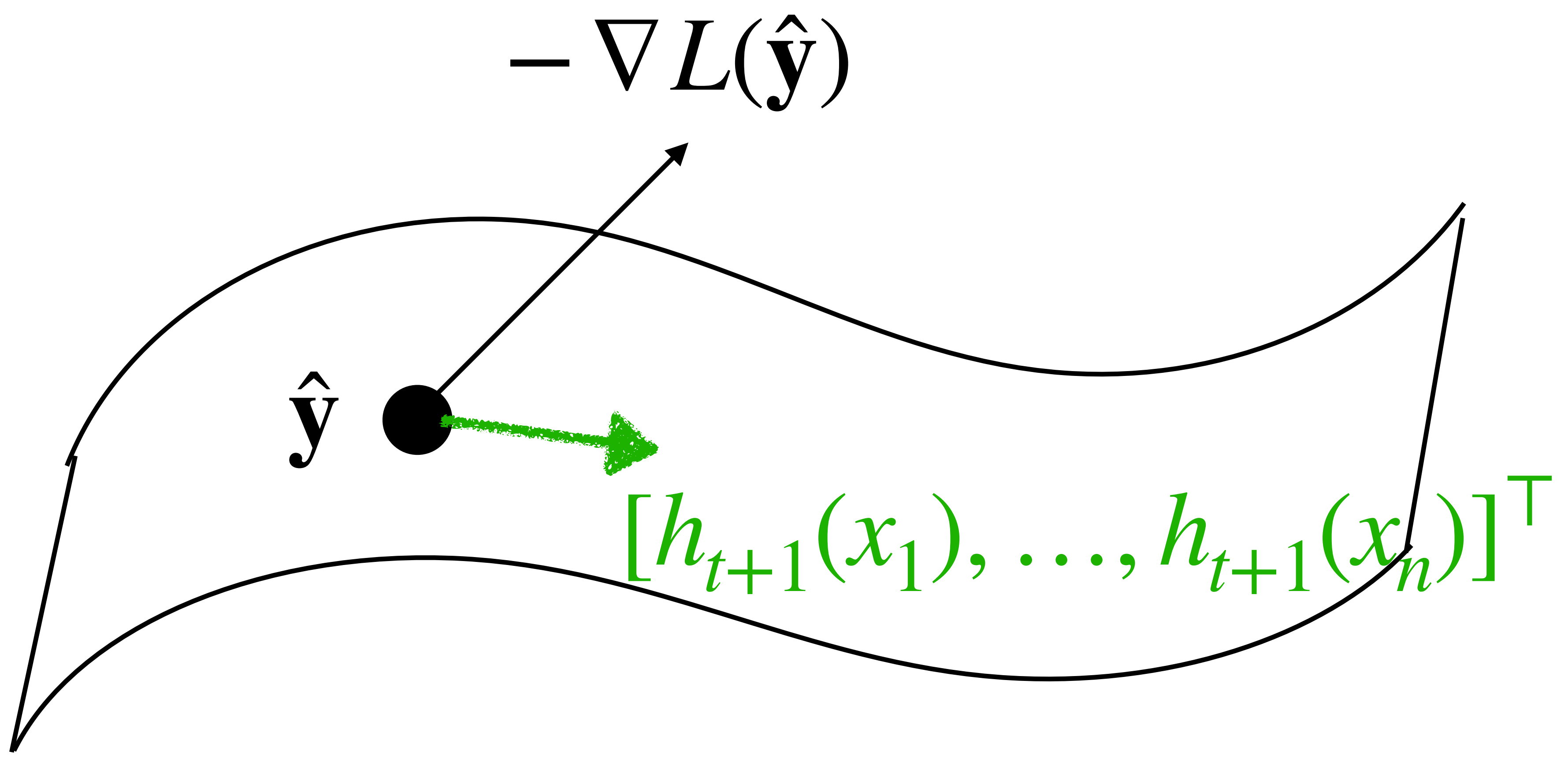

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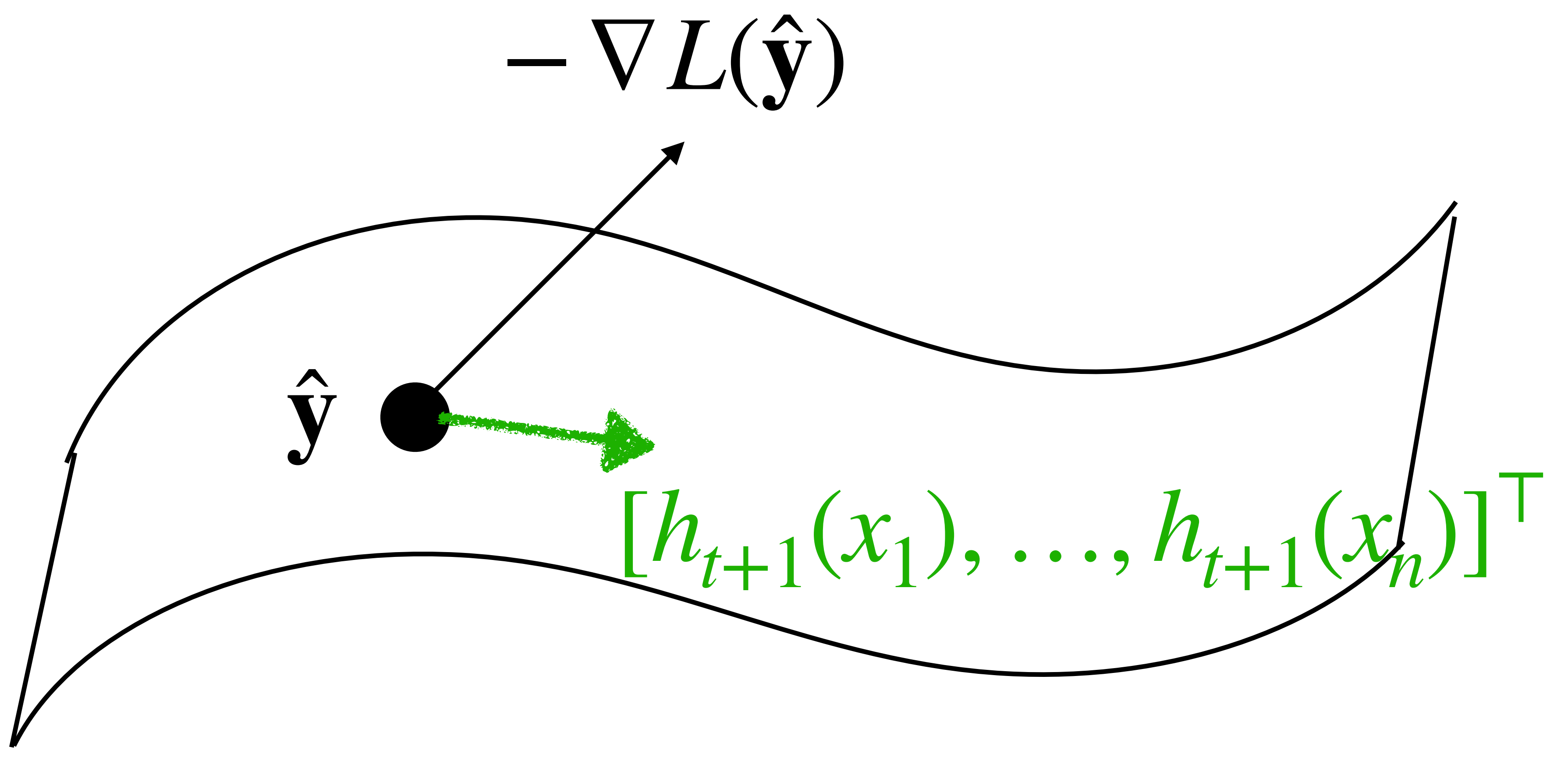

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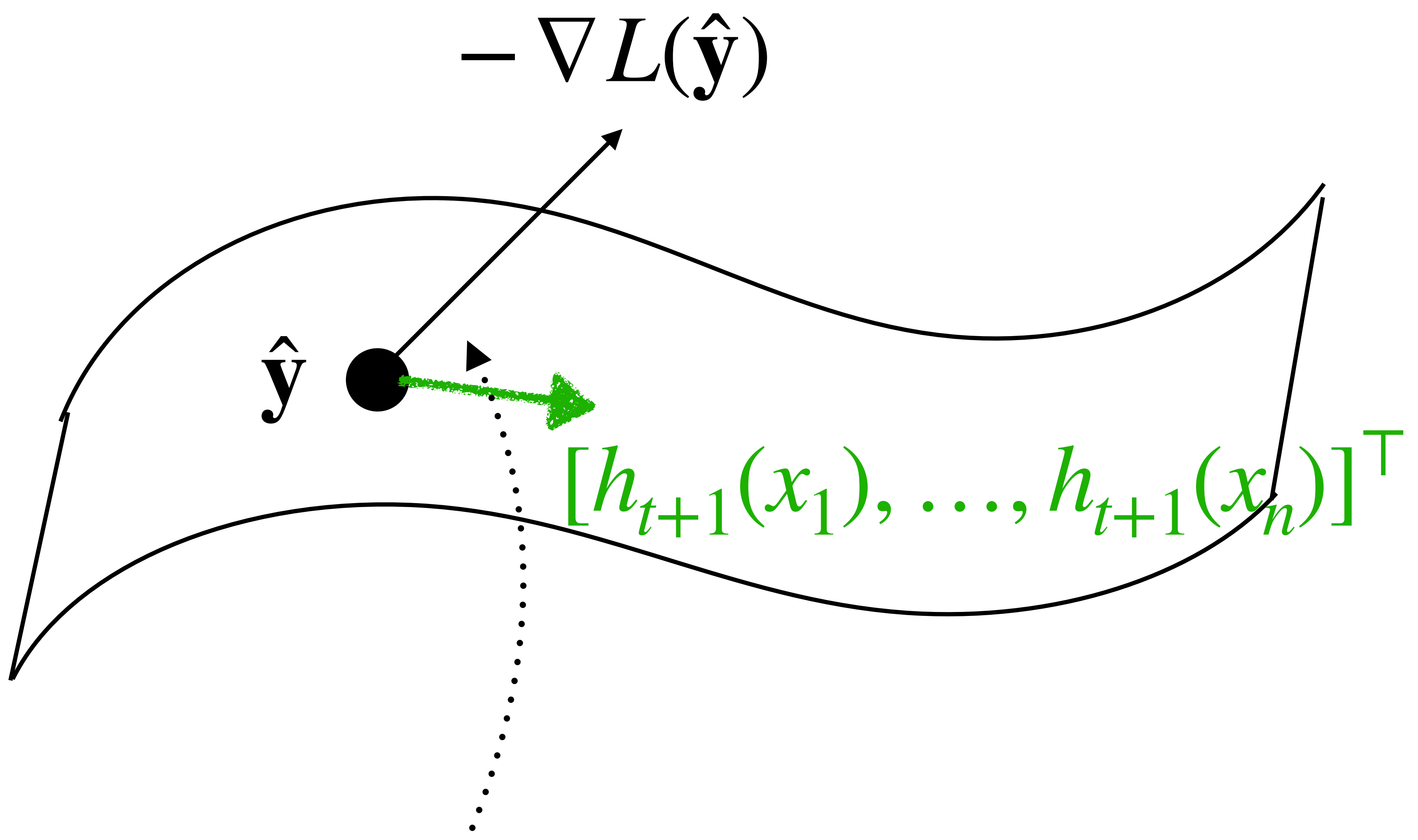

Within 90 degree, so
improve the objective!

Formal Convergence of AdaBoost

Then after T iterations, for the original exp loss, we have

$$\frac{1}{n}\sum_{i=1}^{n}\exp(-H_T(x_i)\cdot y_i) \leq n(1-4\gamma^2)^{T/2}$$

(Proof in lecture note, optional)

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1. Every iteration, we train a weak learner via binary classification on a weighted data

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2. Each weaker learner doing better than random coin toss $(0.5 - \gamma)$ implies stronger learner at the end