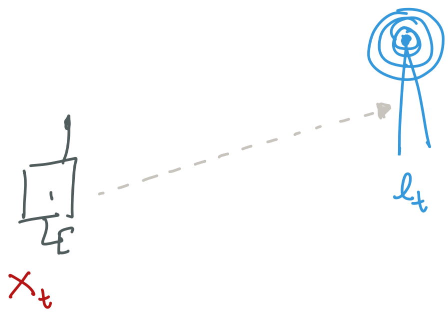
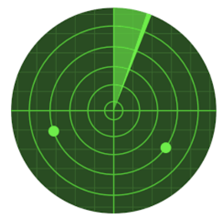
What if Observation Models are Non-Linear?





Example: Radar

$$Z_t = \begin{bmatrix} \text{Distance to landmark} \\ \\ \text{Angle to landmark} \end{bmatrix}$$

$$Z_t = \begin{bmatrix} \sqrt{(x_t^1 - l_t^1)^2 + (x_t^2 - l_t^2)^2} \\ \\ \tan^{-1}\left(\dfrac{l_t^2 - y_t^2}{l_t^1 - x_t^1}\right) \end{bmatrix}$$

$x_t = \begin{bmatrix} x_t^1 \\ x_t^2 \end{bmatrix}$ $l_t = \begin{bmatrix} l_t^1 \\ l_t^2 \end{bmatrix}$

$$= h(x_t)$$

Non-linear function

Observation Model

NOT LINEAR

$$P(z_t \mid x_t) \propto \exp\left(- \frac{\| z_t - h(x_t) \|^2}{\Sigma_t^{obs}} \right)$$

Lets keep prior $P(x_0) \propto \exp\left(- \frac{\| x_0 - \mu_0 \|^2}{\Sigma_t} \right)$

$T = 0$

$$\begin{tikzcd} x_0 \arrow[d] \\ z_0 \end{tikzcd}$$

$$\underset{x_0}{\operatorname{argmax}} \; P(x_0 \mid z_0)$$

$$\underset{x_0}{\operatorname{argmax}} \; \log P(z_0 \mid x_0) \, P(x_0)$$

$$\underset{x_0}{\operatorname{argmax}} \; \log\left(\frac{1}{\eta} \exp\left(- \frac{(z_0 - h(x_0))^2}{\Sigma_0^{obs}} \right) \right) + \log\left(\frac{1}{\eta} \exp\left(- \frac{(x_0 - \mu_0)^2}{\Sigma_0} \right) \right)$$

NON LINEAR !!

$$\underset{x_0}{\operatorname{argmin}} \; \frac{(z_0 - h(x_0))^2}{\Sigma_0^{obs}} + \frac{(x_0 - \mu_0)^2}{\Sigma_0}$$

HOW DO YOU CONVERT SOMETHING NON-LINEAR TO LINEAR?

TAYLOR SERIES ABOUT AN INITIAL GUESS $\bar{X}$

$$h(X) = h(\bar{X}) + \left.\frac{\partial h}{\partial x}\right|_{\bar{X}} (X - \bar{X})$$

$$= \bar{h} + H \cdot \delta x$$

(LINEAR!)

$$\underset{X_0}{\arg\min} \frac{(z_0 - h(X_0))^2}{\Sigma_0^{obs}} + \frac{(X_0 - \mu_0)^2}{\Sigma_0}$$

$$\underset{\delta x}{\arg\min} \frac{(Z_0 - \bar{h}_0 - H\delta x)^2}{\Sigma_0^{obs}} + \frac{(\overbrace{X_0 - \bar{X}_0}^{\delta x} + \bar{X}_0 - \mu_0)^2}{\Sigma_0}$$

$$\underset{\delta x}{\arg\min} \frac{((z_0 - \bar{h}_0) - H\delta x)^2}{\Sigma_0^{obs}} + \frac{(\delta x + (\bar{X}_0 - \mu_0))^2}{\Sigma_0}$$

$$\nabla_{\delta x}(\cdot) = 0 \Rightarrow$$

$$\frac{H^T\left(H\delta x - (z_0 - \bar{h}_0)\right)}{\Sigma_0^{obs}} + \frac{\left(\delta x + (\bar{x}_0 - \mu_0)\right)}{\Sigma_0} = 0$$

$$H^T \Sigma_0^{obs^{-1}}\left(H\delta x - (z_0 - \bar{h}_0)\right) + \Sigma_0^{-1}\left(\delta x + (\bar{x}_0 - \bar{\mu}_0)\right) = 0$$

$$\left(H^T \Sigma_0^{obs^{-1}} H + \Sigma_0^{-1}\right)\delta x = H^T \Sigma_0^{obs^{-1}}(z_0 - \bar{h}_0) + \Sigma_0^{-1}(\bar{\mu}_0 - \bar{x}_0)$$

$$\delta x = \frac{H^T \Sigma_0^{obs^{-1}}(z_0 - \bar{h}_0) + \Sigma_0^{-1}(\bar{\mu}_0 - \bar{x}_0)}{\left(H^T \Sigma_0^{obs^{-1}} H + \Sigma_0^{-1}\right)}$$

SENSITIVITY OF OBSERVATION

THE GAUSS-NEWTON ALGORITHM

INPUT: Z (OBSERVATION), Σ^{obs}, μ : PRIOR, Σ

$\bar{X} \leftarrow$ INITIALIZE GUESS FOR X

WHILE NOT CONVERGED:

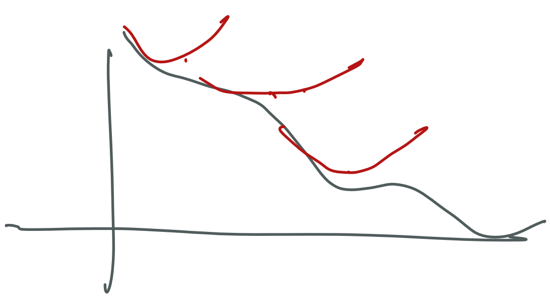
$\bar{h}, H \leftarrow \text{LINEARIZE}(h, \bar{X})$

$\delta x \leftarrow \text{LEAST SQUARES}(\bar{h}, H, Z, \mu)$

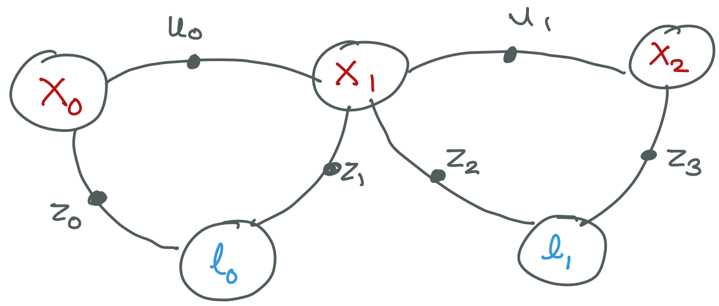
$$\delta x = \frac{H^T {\Sigma_o^{obs}}^{-1}(z_o - \bar{h}_o) + \Sigma_o^{-1}(\mu_o - \bar{x}_o)}{\left(H^T {\Sigma_o^{obs}}^{-1} H + \Sigma_o^{-1}\right)}$$

$\bar{X} \leftarrow \bar{X} - \alpha\, \delta x$

STEP SIZE



Putting Everything Together: Factor Graphs.



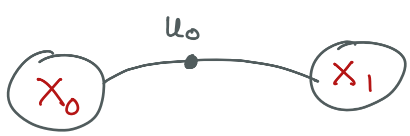
Bipartite Graph

Variable Nodes: x_0, x_1, x_2, l_0, l_1

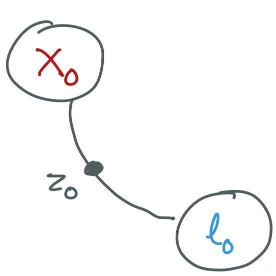
Factor Node: z_0, z_1, z_2, z_3, u_0, u_1

$\boxed{\text{Edges connect variable nodes to factor nodes}}$

Every edge corresponds to a Loss



$$\Rightarrow \left\| u_0 - g\left(x_0, x_1\right) \right\|^2_{\Sigma_0}$$



$$\Rightarrow \left\| z_0 - h\left(x_0, l_0\right) \right\|^2_{\Sigma^{obs}}$$

FACTOR GRAPH SOLVE THE FOLLOWING PROBLEM

$$\underset{\underbrace{x_0,\ x_1,\ x_2,\ l_0,\ l_1}_{\theta}}{\operatorname{argmax}} \log P\Big(\overbrace{x_0,\ x_1,\ x_2,\ l_0,\ l_1}^{\text{VARIABLES}} \,\Big|\, \overbrace{\underbrace{z_0,\ z_1,\ z_2,\ z_3,\ u_0,\ u_1}_{y}}^{\text{OBSERVATION}}\Big)$$

$$\Downarrow$$

$$\underset{\underbrace{x_0,\ x_1,\ x_2,\ l_0,\ l_1}_{\theta}}{\operatorname{argmin}} \sum \| y_i - f_i(\theta) \|_{\Sigma}^{2}$$

$$\Downarrow$$

GAUSS NEWTON

$$\sum \| \underbrace{y_i - \bar{f}_i}_{b} - F\,\delta\theta \|_{\Sigma}^{2}$$

$$F^T F\,\delta\theta = b$$

$$\boxed{\delta\theta = (F^T F)^{-1} b}$$

Gauss Newton for Factor Graph

Input: All observations: $z_0, z_1, \ldots, u_0, u_1, \ldots$

All models: $f(\cdot)$

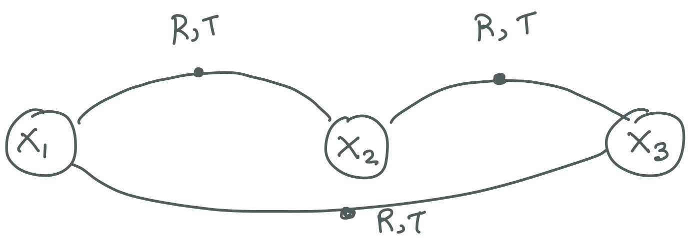
$\bar{\theta} \leftarrow$ Initialize guess $x_0, x_1, \ldots, l_0, l_1$

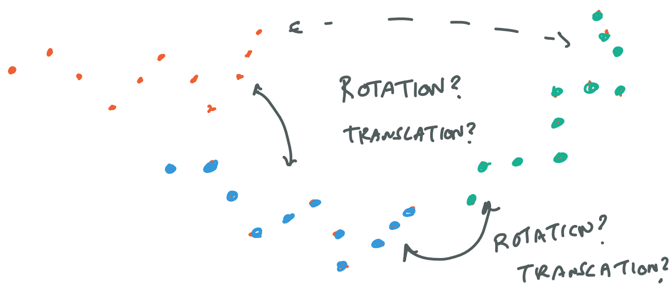
WHILE NOT CONVERGED

$\bar{f}_i, F \leftarrow$ Linearize all factors.

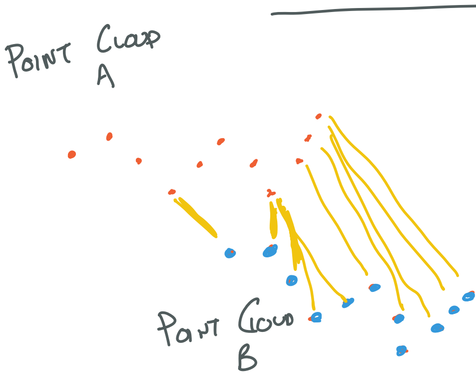
$\delta\theta \leftarrow$ Least Squares $\left(\bar{f}_i, F, z_{1:T}, u_{1:T}, \theta\right)$

$\bar{\theta} \leftarrow \bar{\theta} - \alpha\,\delta\theta$





ITERATIVE CLOSEST POINT



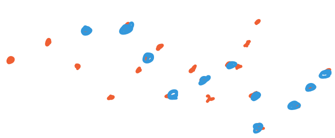
STEP 1:

FOR EVERY POINT IN B, FIND CLOSEST POINT IN A

"CORRESPONDENCE"

STEP 2:

FIND A TRANSLATION SO DISTANCE IS MINIMIZED



REPEAT

STEP 3:

FIND A ROTATION SO DISTANCE IS MINIMIZED