# Reinforcement Learning

**CS472/CS473 – Fall 2005**

## Reinforcement Learning

- **Supervised Learning:**
  - Training examples: (x,y)
  - Direct feedback y for each input x
- **Reinforcement Learning**
  - Sequence of decisions with eventual feedback
  - No teacher that critiques individual actions
  - Learn to act and to assign blame/credit to individual actions
  - Examples
    - when playing a game, only after many actions final result: win, loss, or draw.
    - Robot fetching bagels from bakery
    - Navigating the Web for collecting all CS pages
    - Control problems (reactor control)

## Reinforcement Learning

- **Issues**
  - Agent knows the full environment a priori vs. unknown environment
  - Agent can be passive (watch) or active (explore)
  - Feedback (i.e. rewards) in terminal states only; or a bit of feedback in any state
  - How to measure and estimate the utility of each action
  - Environment fully observable, or partially observable
  - Have model of environment and effects of action…or not

→ **Reinforcement Learning will adress these issues!**

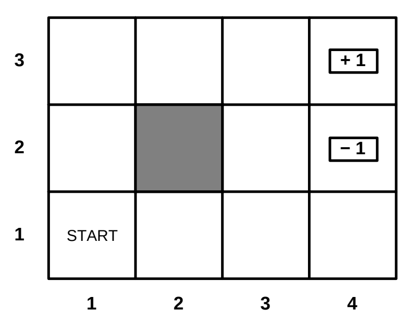
## Markov Decision Process

- **Representation of Environment:**
  - finite set of states S
  - set of actions A for each state $s \in S$
- **Process**
  - At each discrete time step, the agent
    - observes state $s_t \in S$ and then
    - chooses action $a_t \in A$.
  - After that, the environment
    - gives agent an immediate reward $r_t$
    - changes state to $s_{t+1}$ (can be probabilistic)

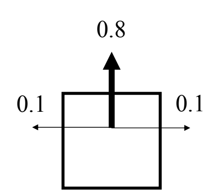
## Markov Decision Process

- **Model:**
  - Initial state: $S_0$
  - Transition function: T(s,a,s')
    → T(s,a,s') is the probability of moving from state s to s' when executing action a.
  - Reward function: R(s)
    → Real valued reward that the agent receives for entering state s.
- **Assumptions**
  - Markov property: T(s,a,s') and R(s) only depend on current state s, but not on any states visited earlier.
  - Extension: Function R may be non-deterministic as well

## Example

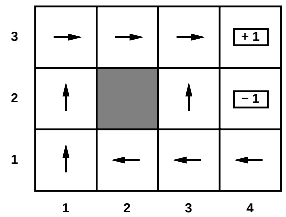



Each other state has a reward of -0.04.



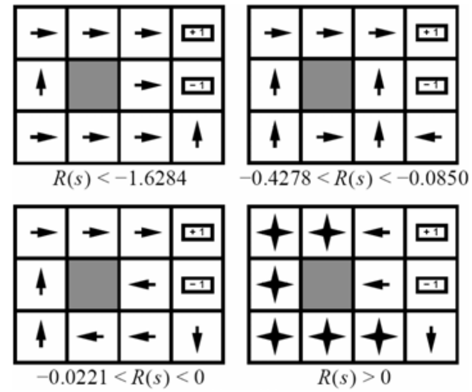

- move into desired direction with prob 80%
- move 90 degrees to left with prob 10%
- move 90 degrees to right with prob 10%

# Policy




- **Definition:**
  - A policy $\pi$ describes which action an agent selects in each state
  - a=$\pi$(s)
- **Utility**
  - For now: $U([s_0,\ldots,s_N]) = \Sigma_i R(s_i)$
  - Let $P([s_0,\ldots,s_N] \mid \pi, s_0)$ be the probability of state sequence $[s_0,\ldots,s_N]$ when following policy $\pi$ from state $s_0$
  - Expected utility: $U^\pi(s) = \Sigma\, U([s_0,\ldots,s_N])\, P([s_0,\ldots,s_N] \mid \pi, s_0)$
    → measure of quality of policy $\pi$
  - Optimal policy $\pi^*$: Policy with maximal $U^\pi(s)$ in each state s

# Optimal Policies for Other Rewards

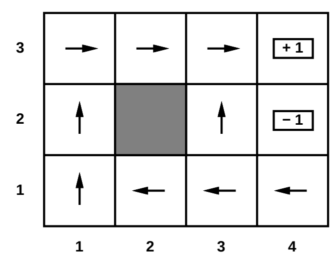



# Utility (revisited)

- **Problem:**
  - What happens to utility value when
    - either the state space has no terminal states
    - or the policy never directs the agent to a terminal state
  
    → Utility becomes infinite
- **Solution**
  - Discount factor $0 < \gamma < 1$
    → closer rewards count more than awards far in the future
  - $U([s_0,\ldots,s_N]) = \Sigma_i\, \gamma^i\, R(s_i)$
    → finite utility even for infinite state sequences

# How to Compute the Utility for a given Policy?

- **Definition: $U^\pi(s) = \Sigma\, [\, \Sigma_i\, \gamma^i\, R(s_i)\, ]\, P([s_0, s_1,\ldots] \mid \pi, s_0{=}s)$**
- **Recursive computation:**
  - $U^\pi(s) = R(s) + \gamma\, \Sigma_{s'}\, T(s, \pi(s), s')\, U^\pi(s')$




| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 3 | 0.812 | 0.868 | 0.918 | +1 |
| 2 | 0.762 | | 0.660 | −1 |
| 1 | 0.705 | 0.655 | 0.611 | 0.388 |

Here: $\gamma$=1.0, R(s)=-0.04

# Bellman Update (for fixed $\pi$)

**Goal: Solve set of n=|S| equations (one for each state)**

$U^\pi(s_0) = R(s_0) + \gamma\, \Sigma_{s'}\, T(s_0, \pi(s), s')\, U^\pi(s')$

…

$U^\pi(s_n) = R(s_n) + \gamma\, \Sigma_{s'}\, T(s_n, \pi(s), s')\, U^\pi(s')$

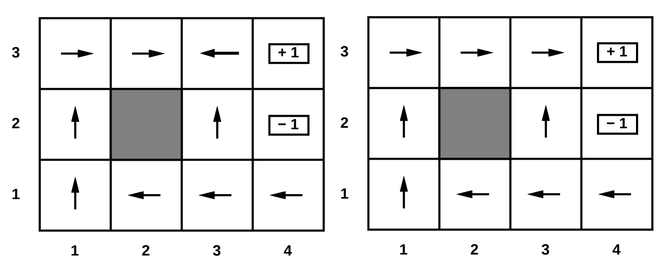
**Algorithm [Policy Evaluation]:**

- i=0; $U^\pi_0(s)$=0 for all s
- repeat
  - i = i +1
  - for each state s in S do
    - $U^\pi_i(s) = R(s) + \gamma\, \Sigma_{s'}\, T(s, \pi(s), s')\, U^\pi_{i-1}(s')$
  - endfor
- until difference between $U^\pi_i$ and $U^\pi_{i-1}$ small enough
- return $U^\pi_i$

# How to Find the Optimal Policy $\pi^*$?

**Is policy $\pi$ optimal? How can we tell?**

- If $\pi$ is not optimal, then there exists some state where
  $\pi(s) \neq \text{argmax}_a\, \boldsymbol{\Sigma_{s'}\, T(s, a, s')\, U^\pi(s')}$
- How to find the optimal policy $\pi^*$?



# How to Find the Optimal Policy $\pi^*$?

**Algorithm [Policy Iteration]:**

- repeat
  - $U^\pi$ = PolicyEvaluation($\pi$,S,T,R)
  - for each state s in S do
    - If [ $\max_a \Sigma_{s'}$ **T(s, a, s') $U^\pi$(s')** > $\Sigma_{s'}$ **T(s, $\pi$(s), s') $U^\pi$(s') ] then**
      - » $\pi$(s) = $\text{argmax}_a \Sigma_{s'}$ **T(s, a, s') $U^\pi$(s')**
  - **endfor**
- until $\pi$ does not change any more
- return $\pi$

# Utility ⇔ Policy

**Equivalence:**

- If we know the optimal utility U(s) of each state, we can derive the optimal policy:

  $\pi^*(s) = \text{argmax}_a \Sigma_{s'}$ **T(s, a, s') U(s')**

- If we know the optimal policy $\pi^*$, we can compute the optimal utility of each state:

  PolicyEvaluation algorithm

**Bellman Equation:**

$$U(s) = R(s) + \gamma \max_a \Sigma_{s'} T(s, a, s')\, U(s')$$

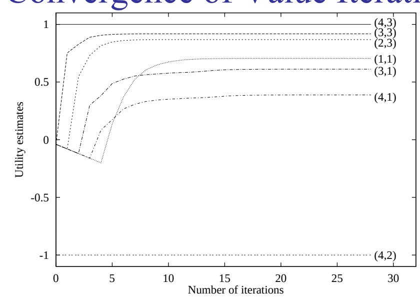
→ Necessary and sufficient condition for optimal U(s).

# Value Iteration Algorithm

- **Algorithm [Value Iteration]:**
  - i=0; $U_0$(s)=0 for all s
  - repeat
    - i = i +1
    - for each state s in S do
      - $U_i(s) = R(s) + \gamma \max_a \Sigma_{s'} T(s, a, s')\, U_{i-1}(s')$
    - endfor
  - until difference between $U_i$ and $U_{i-1}$ small enough
  - return $U_i$

→ derive optimal policy via $\pi^*(s) = \text{argmax}_a \Sigma_{s'} T(s, a, s')\, U(s')$

# Convergence of Value Iteration




- **Value iteration is guaranteed to converge to optimal U for $0 \le \gamma < 1$**
- **Faster convergence for smaller $\gamma$**

# Reinforcement Learning

**Assumptions we made so far:**

- Known state space S
- Known transition model T(s, a, s')
- Known reward function R(s)

→not realistic for many real agents

**Reinforcement Learning:**

- Learn optimal policy with a priori unknown environment
- Assume fully observable environment (i.e. agent can tell it's state)
- Agent needs to explore environment (i.e. experimentation)

# Passive Reinforcement Learning

**Task: Given a policy $\pi$, what is the utility function $U^\pi$ ?**

- Similar to Policy Evaluation, but unknown T(s, a, s') and R(s)

**Approach: Agent experiments in the environment**

- Trials: execute policy from start state until in terminal state.

$(1,1)_{-0.04} \rightarrow (1,2)_{-0.04}$
$\rightarrow (1,3)_{-0.04} \rightarrow (1,2)_{-0.04}$
$\rightarrow (1,3)_{-0.04} \rightarrow (2,3)_{-0.04}$
$\rightarrow (3,3)_{-0.04} \rightarrow (4,3)_{1.0}$

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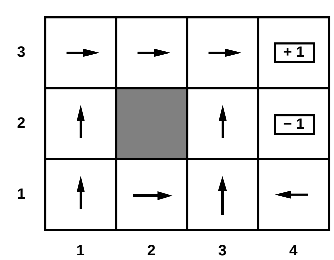

## Direct Utility Estimation

- **Data: Trials of the form**
  - $(1,1)_{-0.04} \rightarrow (1,2)_{-0.04} \rightarrow (1,3)_{-0.04} \rightarrow (1,2)_{-0.04} \rightarrow (1,3)_{-0.04} \rightarrow (2,3)_{-0.04} \rightarrow (3,3)_{-0.04} \rightarrow (4,3)_{1.0}$
  - $(1,1)_{-0.04} \rightarrow (1,2)_{-0.04} \rightarrow (1,3)_{-0.04} \rightarrow (2,3)_{-0.04} \rightarrow (3,3)_{-0.04} \rightarrow (3,2)_{-0.04} \rightarrow (3,3)_{-0.04} \rightarrow (4,3)_{1.0}$
  - $(1,1)_{-0.04} \rightarrow (2,1)_{-0.04} \rightarrow (3,1)_{-0.04} \rightarrow (3,2)_{-0.04} \rightarrow (4,2)_{-1.0}$
- **Idea:**
  - Average reward over all trials for each state independently
    - ➔Supervised Learning Problem
- **Why is this less efficient than necessary?**
  - ➔ Ignores dependencies between states

$$U^\pi(s) = R(s) + \gamma\, \Sigma_{s'}\, T(s, \pi(s), s')\, U^\pi(s')$$

## Adaptive Dynamic Programming (ADP)

- **Idea:**
  - Run trials to learn model of environment (i.e. T and R)
    - Memorize R(s) for all visited states
    - Estimate fraction of times action a from state s leads to s'
  - Use PolicyEvaluation Algorithm on estimated model
- **Problem?**
  - Can be quite costly for large state spaces
  - For example, Backgammon has $10^{50}$ states
    - ➔ Learn and store all transition probabilities and rewards
    - ➔ PolicyEvaluation needs to solve linear program with $10^{50}$ equations and variables.

## Temporal Difference (TD) Learning

- **Idea:**
  - Do not learn explicit model of environment!
  - Use update rule that implicitly reflects transition probabilities.
- **Method:**
  - Init $U^\pi(s)$ with R(s) when first visited
  - After each transition, update with
    $$U^\pi(s) = U^\pi(s) + \alpha\, [R(s) + \gamma\, U^\pi(s') - U^\pi(s)]$$
  - $\alpha$ is learning rate. $\alpha$ should decrease slowly over time, so that estimates stabilize eventually.
- **Properties:**
  - No need to store model
  - Only one update for each action (not full PolicyEvaluation)

**Data:**

- $(1,1)_{-0.04} \rightarrow (1,2)_{-0.04} \rightarrow (1,3)_{-0.04} \rightarrow (1,2)_{-0.04} \rightarrow (1,3)_{-0.04} \rightarrow (2,3)_{-0.04} \rightarrow (3,3)_{-0.04} \rightarrow (4,3)_{1.0}$
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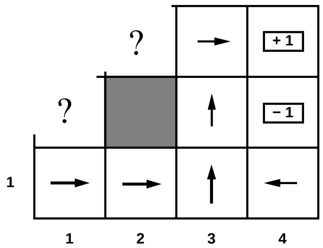
## Active Reinforcement Learning

- **Task: In an a priori unknown environment, find the optimal policy.**
  - unknown T(s, a, s') and R(s)
  - Agent must experiment with the environment.
- **Naïve Approach: "Naïve Active PolicyIteration"**
  - Start with some random policy
  - ~~Follow policy~~ to learn model of environment and use ADP to estimate utilities.
  - Update policy using $\pi(s) \leftarrow \text{argmax}_a\, \Sigma_{s'}\, T(s, a, s')\, U^\pi(s')$
- **Problem:**
  - Can converge to sub-optimal policy!
  - By following policy, agent might never learn T and R everywhere.
    - ➔ Need for exploration!

## Exploration vs. Exploitation

- **Exploration:**
  - Take actions that explore the environment
  - Hope: possibly find areas in the state space of higher reward
  - Problem: possibly take suboptimal steps
- **Exploitation:**
  - Follow current policy
  - Guaranteed to get certain expected reward
- **Approach:**
  - Sometimes take random steps
  - Bonus reward for states that have not been visited often yet




## Q-Learning

- **Problem: Agent needs model of environment to select action via**

$$\text{argmax}_a\, \Sigma_{s'}\, T(s, a, s')\, U^\pi(s')$$

- **Solution: Learn action utility function Q(a,s), not state utility function U(s). Define Q(a,s) as**

$$U(s) = \max_a Q(a,s)$$

➔Bellman equation with Q(a,s) instead of U(s)

$$Q(a,s) = R(s) + \gamma\, \Sigma_{s'}\, T(s, a, s')\, \max_{a'} Q(a',s')$$

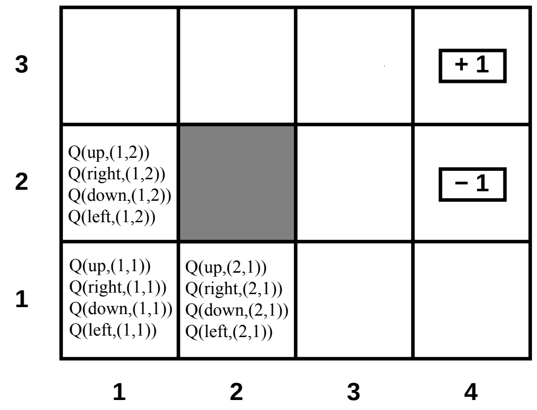
➔TD-Update with Q(a,s) instead of U(s)

$$Q(a,s) \leftarrow Q(a,s) + \alpha\, [R(s) + \gamma\, \max_{a'} Q(a',s') - Q(a,s)]$$

- **Result: With Q-function, agent can select action without model of environment**

$$\text{argmax}_a\, Q(a,s)$$

# Q-Learning Illustration




# Function Approximation

- **Problem:**
  - Storing Q or U,T,R for each state in a table is too expensive, if number of states is large
  - Does not exploit "similarity" of states (i.e. agent has to learn separate behavior for each state, even if states are similar)
- **Solution:**
  - Approximate function using parametric representation
  - For example: $U(s) = \vec{w} \cdot \Phi(s)$
    - $\Phi(s)$ is feature vector describing the state
      - "Material values" of board
      - Is the queen threatened?
      - …