CS4670/5760: Computer Vision

Kavita Bala

Lecture 12: Image alignment

http://www.wired.com/gadgetlab/2010/07/camera-software-lets-you-see-into-the-past/

Announcements

- Project 1 Artifact voting closes tomorrow. Please vote.

- Project 2 released
  - **Due Monday, March 9 2015 (by 11:59pm)**

- Prelim next Thu
  - Send me mail if you have a conflict
  - All material till end of this week
  - Closed book
  - Will release last year's midterm. But it was a take home.

PA2:
Feature detection and matching
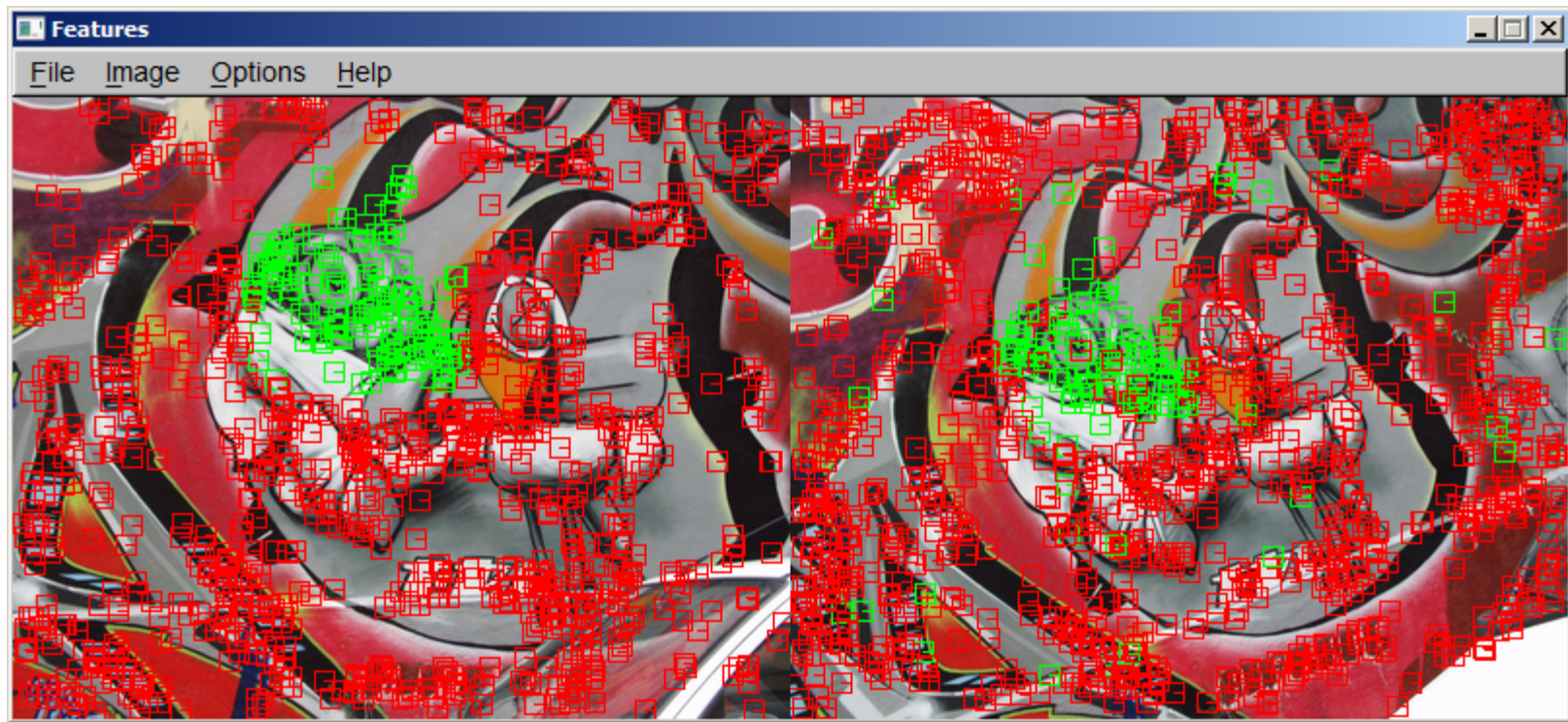
Reading

- Appendix A.2, 6.1

Common problems in vision

Fitting: find the parameters of a model that best fit the data

Alignment: find the parameters of the transformation that best align matched points
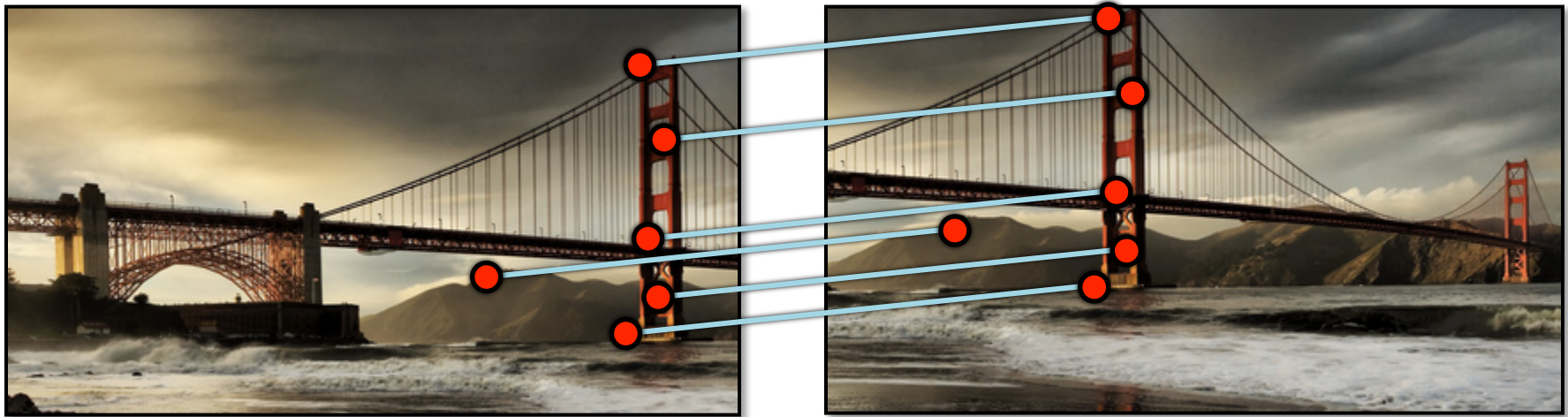
Alignment

- Alignment: find parameters of model that maps one set of points to another

- Typically want to solve for a global transformation that accounts for *most* true correspondences

- Difficulties
  - Noise (typically 1-3 pixels)
  - Outliers (often 50%)
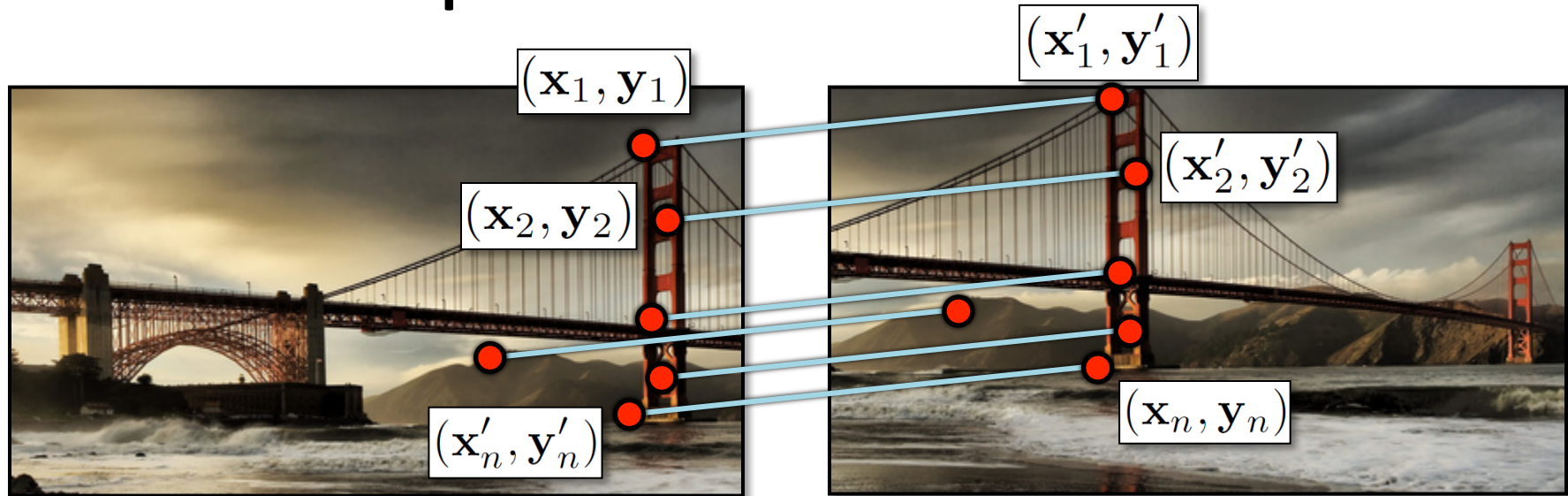
Computing transformations
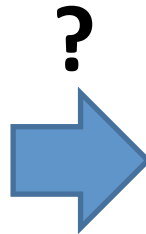
Simple case: translations

How do we solve for $(\mathbf{x}_t, \mathbf{y}_t)$**?**
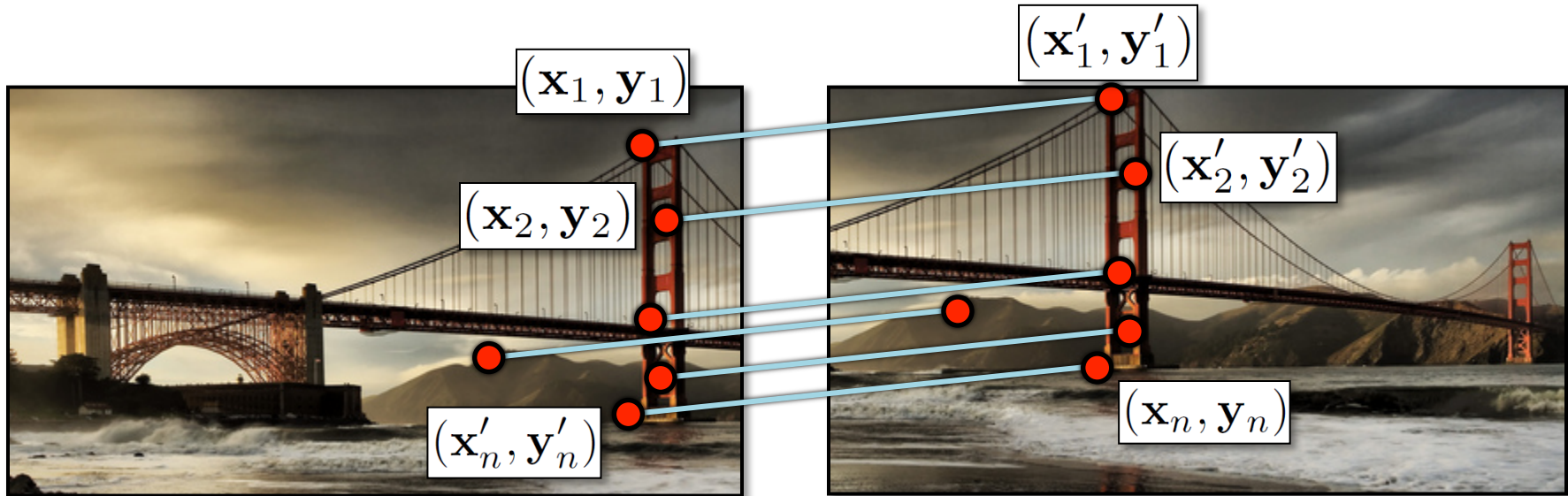
$(\mathbf{x}_t, \mathbf{y}_t)$

Simple case: translations

Displacement of match i = $\left(\mathbf{x}'_i - \mathbf{x}_i, \mathbf{y}'_i - \mathbf{y}_i\right)$

$$\left(\mathbf{x}_t, \mathbf{y}_t\right) = \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}'_i - \mathbf{x}_i, \frac{1}{n}\sum_{i=1}^{n}\mathbf{y}'_i - \mathbf{y}_i\right)$$
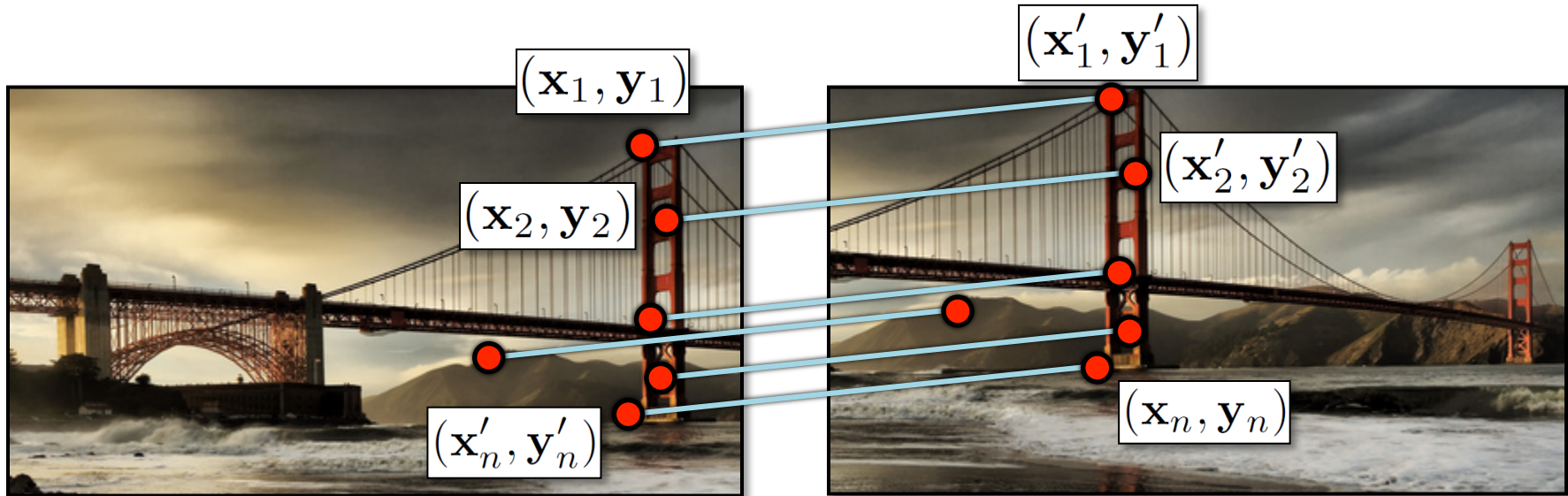
Another view

$$\mathbf{x}_i + \mathbf{x_t} = \mathbf{x}'_i$$

$$\mathbf{y}_i + \mathbf{y_t} = \mathbf{y}'_i$$

- System of linear equations
  - What are the knowns? Unknowns?
  - How many unknowns? How many equations (per match)?

Another view

$$\mathbf{x}_i + \mathbf{x_t} = \mathbf{x}'_i$$

$$\mathbf{y}_i + \mathbf{y_t} = \mathbf{y}'_i$$

- Problem: more equations than unknowns
  - "Overdetermined" system of equations
  - We will find the *least squares* solution

Least squares formulation

- For each point $(\mathbf{x}_i, \mathbf{y}_i)$

$$\mathbf{x}_i + \mathbf{x_t} = \mathbf{x}_i'$$

$$\mathbf{y}_i + \mathbf{y_t} = \mathbf{y}_i'$$

- we define the *residuals* as

$$r_{\mathbf{x}_i}(\mathbf{x}_t) = (\mathbf{x}_i + \mathbf{x}_t) - \mathbf{x}_i'$$

$$r_{\mathbf{y}_i}(\mathbf{y}_t) = (\mathbf{y}_i + \mathbf{y}_t) - \mathbf{y}_i'$$

Least squares formulation

- Goal: minimize sum of squared residuals

$$C(\mathbf{x}_t, \mathbf{y}_t) = \sum_{i=1}^{n} \left(r_{\mathbf{x}_i}(\mathbf{x}_t)^2 + r_{\mathbf{y}_i}(\mathbf{y}_t)^2 \right)$$

- "Least squares" solution
- For translations, is equal to mean displacement

Least squares formulation

- Can also write as a matrix equation

$$
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
1 & 0 \\
0 & 1 \\
& \vdots \\
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x_t \\
y_t
\end{bmatrix}
=
\begin{bmatrix}
x_1' - x_1 \\
y_1' - y_1 \\
x_2' - x_2 \\
y_2' - y_2 \\
\vdots \\
x_n' - x_n \\
y_n' - y_n
\end{bmatrix}
$$

$$\mathbf{A} \quad \mathbf{t} = \mathbf{b}$$

$2n$ x 2 $\qquad$ 2 x 1 $\qquad$ $2n$ x 1

Least squares

$$\mathbf{At} = \mathbf{b}$$

- Find **t** that minimizes

$$||\mathbf{At} - \mathbf{b}||^2$$

Least squares: find t to minimize

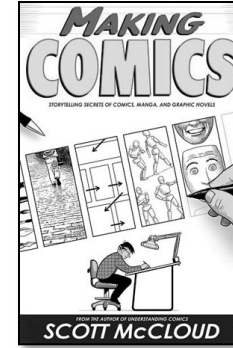$$||\mathbf{At} - \mathbf{b}||^2$$

$$t^T(A^T A)t - 2t^T(A^T b) + ||b||^2$$

- To solve, form the *normal equations*
    - Differentiate and equate to 0 to minimize

$$\mathbf{A}^T\mathbf{At} = \mathbf{A}^T\mathbf{b}$$

$$\mathbf{t} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$$

Affine transformations

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- How many unknowns?

- How many equations per match?

- How many matches do we need?

Affine transformations

- Residuals:

$$r_{x_i}(a, b, c, d, e, f) = (ax_i + by_i + c) - x'_i$$
$$r_{y_i}(a, b, c, d, e, f) = (dx_i + ey_i + f) - y'_i$$

- Cost function:

$$C(a, b, c, d, e, f) =$$
$$\sum_{i=1}^{n} \left(r_{x_i}(a, b, c, d, e, f)^2 + r_{y_i}(a, b, c, d, e, f)^2 \right)$$
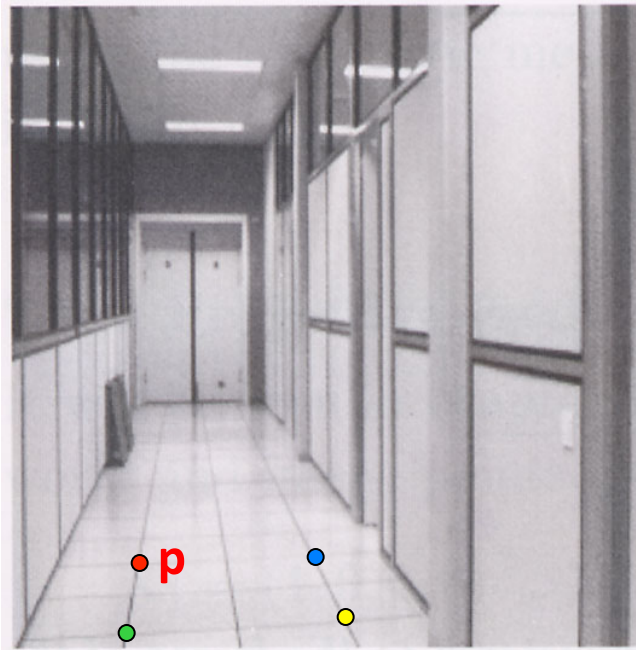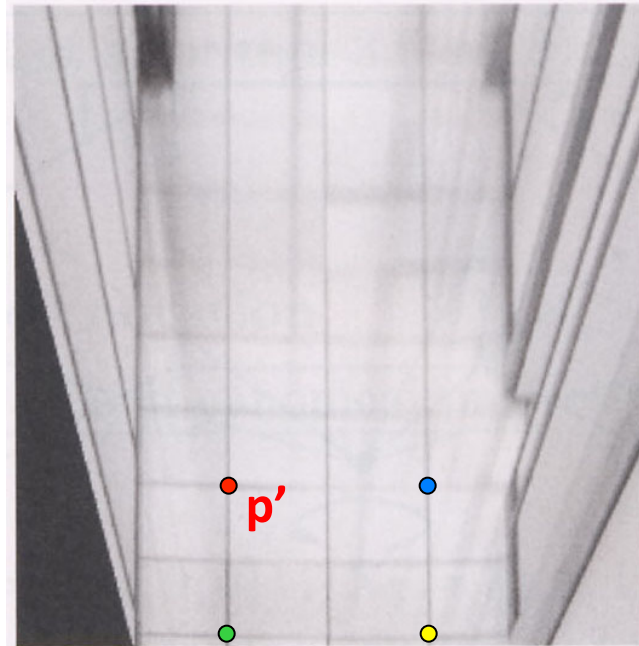
Affine transformations

- Matrix form

$$
\begin{bmatrix}
x_1 & y_1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & x_1 & y_1 & 1 \\
x_2 & y_2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & x_2 & y_2 & 1 \\
& & \vdots & & & \\
x_n & y_n & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & x_n & y_n & 1
\end{bmatrix}
\begin{bmatrix}
a \\ b \\ c \\ d \\ e \\ f
\end{bmatrix}
=
\begin{bmatrix}
x_1' \\ y_1' \\ x_2' \\ y_2' \\ \vdots \\ x_n' \\ y_n'
\end{bmatrix}
$$

$\mathbf{A}$ $\mathbf{t}$ $=$ $\mathbf{b}$

$2n$ x 6   6 x 1 $2n$ x 1

Homographies

To unwarp (rectify) an image

- solve for homography **H** given **p** and **p'**
- solve equations of the form: **p'** = **Hp**

Alternate formulation for homographies

$$\begin{bmatrix} x'_i \\ y'_i \\ 1 \end{bmatrix} \cong \begin{bmatrix} h_{00} & h_{01} & h_{02} \\ h_{10} & h_{11} & h_{12} \\ h_{20} & h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}$$

where the length of the vector $[h_{00}\ h_{01}\ \dots\ h_{22}]$ is 1

Solving for homographies

$$\begin{bmatrix} x'_i \\ y'_i \\ 1 \end{bmatrix} \cong \begin{bmatrix} h_{00} & h_{01} & h_{02} \\ h_{10} & h_{11} & h_{12} \\ h_{20} & h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}$$

$$x'_i = \frac{h_{00}x_i + h_{01}y_i + h_{02}}{h_{20}x_i + h_{21}y_i + h_{22}}$$

$$y'_i = \frac{h_{10}x_i + h_{11}y_i + h_{12}}{h_{20}x_i + h_{21}y_i + h_{22}}$$

Linear or non-linear?

Solving for homographies

$$\begin{bmatrix} x'_i \\ y'_i \\ 1 \end{bmatrix} \cong \begin{bmatrix} h_{00} & h_{01} & h_{02} \\ h_{10} & h_{11} & h_{12} \\ h_{20} & h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}$$

$$x'_i = \frac{h_{00}x_i + h_{01}y_i + h_{02}}{h_{20}x_i + h_{21}y_i + h_{22}}$$

$$y'_i = \frac{h_{10}x_i + h_{11}y_i + h_{12}}{h_{20}x_i + h_{21}y_i + h_{22}}$$

$$x'_i(h_{20}x_i + h_{21}y_i + h_{22}) = h_{00}x_i + h_{01}y_i + h_{02}$$
$$y'_i(h_{20}x_i + h_{21}y_i + h_{22}) = h_{10}x_i + h_{11}y_i + h_{12}$$

Solving for homographies

$$x_i'(h_{20}x_i + h_{21}y_i + h_{22}) = h_{00}x_i + h_{01}y_i + h_{02}$$
$$y_i'(h_{20}x_i + h_{21}y_i + h_{22}) = h_{10}x_i + h_{11}y_i + h_{12}$$

$$\begin{bmatrix} x_i & y_i & 1 & 0 & 0 & 0 & -x_i'x_i & -x_i'y_i & -x_i' \\ 0 & 0 & 0 & x_i & y_i & 1 & -y_i'x_i & -y_i'y_i & -y_i' \end{bmatrix} \begin{bmatrix} h_{00} \\ h_{01} \\ h_{02} \\ h_{10} \\ h_{11} \\ h_{12} \\ h_{20} \\ h_{21} \\ h_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving for homographies

$$
\begin{bmatrix}
x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1' x_1 & -x_1' y_1 & -x_1' \\
0 & 0 & 0 & x_1 & y_1 & 1 & -y_1' x_1 & -y_1' y_1 & -y_1' \\
 & & & & \vdots & & & & \\
x_n & y_n & 1 & 0 & 0 & 0 & -x_n' x_n & -x_n' y_n & -x_n' \\
0 & 0 & 0 & x_n & y_n & 1 & -y_n' x_n & -y_n' y_n & -y_n'
\end{bmatrix}
\begin{bmatrix}
h_{00} \\ h_{01} \\ h_{02} \\ h_{10} \\ h_{11} \\ h_{12} \\ h_{20} \\ h_{21} \\ h_{22}
\end{bmatrix}
=
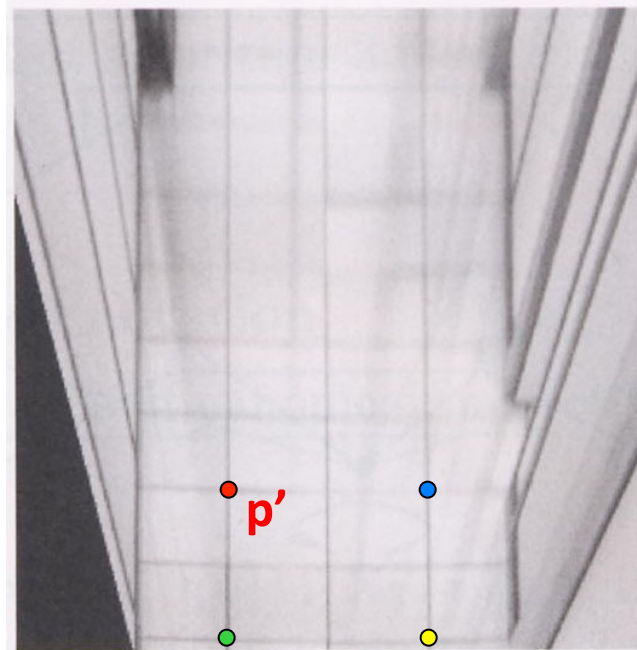\begin{bmatrix}
0 \\ 0 \\ \vdots \\ 0 \\ 0
\end{bmatrix}
$$

$$\mathbf{A}$$
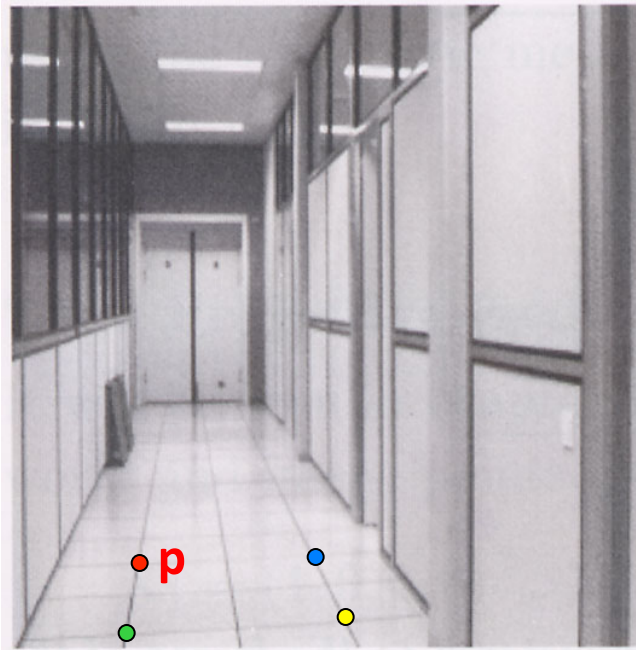**2n × 9**

$$\mathbf{h}$$
9

$$\mathbf{0}$$
2n

Defines a least squares problem: $\text{minimize } \|\mathbf{A}\mathbf{h} - \mathbf{0}\|^2$

- Since $\mathbf{h}$ is only defined up to scale, solve for unit vector $\hat{\mathbf{h}}$

Homographies

To unwarp (rectify) an image

- solve for homography **H** given **p** and **p'**
- solve equations of the form: **p'** = **Hp**
  - linear in unknowns: coefficients of **H**
  - H is defined up to an arbitrary scale factor
  - how many points are necessary to solve for **H**?

Solving for homographies

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -x'_1 x_1 & -x'_1 y_1 & -x'_1 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -y'_1 x_1 & -y'_1 y_1 & -y'_1 \\ & & & & \vdots & & & & \\ x_n & y_n & 1 & 0 & 0 & 0 & -x'_n x_n & -x'_n y_n & -x'_n \\ 0 & 0 & 0 & x_n & y_n & 1 & -y'_n x_n & -y'_n y_n & -y'_n \end{bmatrix} \begin{bmatrix} h_{00} \\ h_{01} \\ h_{02} \\ h_{10} \\ h_{11} \\ h_{12} \\ h_{20} \\ h_{21} \\ h_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{A}$$
2n × 9

$$\mathbf{h}$$
9

$$\mathbf{0}$$
2n

Defines a least squares problem: minimize $\|\mathbf{A}\mathbf{h} - 0\|^2$

- Since $\mathbf{h}$ is only defined up to scale, solve for unit vector $\hat{\mathbf{h}}$
- Solution: $\hat{\mathbf{h}}$ = eigenvector of $\mathbf{A}^T\mathbf{A}$ with smallest eigenvalue
- Works with 4 or more points

Recap: Two Common Optimization Problems

Problem statement	Solution
minimize $\|\mathbf{Ax} - \mathbf{b}\|^2$	$\mathbf{x} = \left(\mathbf{A}^T \mathbf{A}\right)^{-1} \mathbf{A}^T \mathbf{b}$
least squares solution to $\mathbf{Ax} = \mathbf{b}$	$\mathbf{x} = \mathbf{A} \setminus \mathbf{b}$ (matlab)

Problem statement	Solution
minimize $\mathbf{x}^T \mathbf{A}^T \mathbf{A} \mathbf{x}$ s.t. $\mathbf{x}^T \mathbf{x} = 1$	$[\mathbf{v}, \lambda] = \mathrm{eig}(\mathbf{A}^T \mathbf{A})$
	$\lambda_1 < \lambda_{2..n} : \mathbf{x} = \mathbf{v}_1$
non - trivial lsq solution to $\mathbf{Ax} = 0$	

Fig. 2.4. **Removing perspective distortion.** *(a) The original image with perspective distortion – the lines of the windows clearly converge at a finite point. (b) Synthesized frontal orthogonal view of the front wall. The image (a) of the wall is related via a projective transformation to the true geometry of the wall. The inverse transformation is computed by mapping the four imaged window corners to corners of an appropriately sized rectangle. The four point correspondences determine the transformation. The transformation is then applied to the whole image. Note that sections of the image of the ground are subject to a further projective distortion. This can also be removed by a projective transformation.*