CS 465 Solution 2

(revised September 20, 2004)

Problem 2: Ray-Object Intersections Each sphere ray object intersection can be computed by solving the quadratic equation that results from plugging the ray equation into the equation for a sphere. A similar process will yield a the ray triangle intersection location. Finally, the location with the smallest, nonnegative t value is the closest location.

Sphere Equation:

$$a = (\vec{\mathbf{d}} \cdot \vec{\mathbf{d}}), \quad b = [2.0 \cdot \vec{\mathbf{d}} \cdot (\vec{\mathbf{o}} - \vec{\mathbf{c}})], \quad c = [(\vec{\mathbf{o}} - \vec{\mathbf{c}}) \cdot (\vec{\mathbf{o}} - \vec{\mathbf{c}}) - r^2], \quad t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Triangle Equation: $a = \vec{\mathbf{n}} \cdot \vec{\mathbf{d}}, \quad b = \vec{\mathbf{n}} \cdot (\vec{\mathbf{o}} - \vec{\mathbf{v_1}}), \quad t = -\frac{b}{a}$.

For our purposes, $r = 1.25$, $\vec{\mathbf{c}} = [1,1,1]^T$, $\vec{\mathbf{n}} = [1,1,1]^T$, and $v_1 = [1,0,0]^T$.

1. $\vec{\mathbf{o}} = [0,0,0]^T, \vec{\mathbf{d}} = [1,1,1]^T$:

   (a) Sphere Intersection: $a = 3$, $b = -6$, $c = \frac{23}{16}, t = \frac{6 \pm \sqrt{36 - \frac{69}{4}}}{6}$
   $t_1 \approx 1.7217$ $\quad p_1 \approx 1.7217[1,1,1]^T$
   $t_2 \approx 0.2783$ $\quad p_2 \approx 0.2783[1,1,1]^T$

   (b) Triangle Intersection: $a = 3$, $b = -1$
   $t = \frac{1}{3}$ $\quad p = \frac{1}{3}[1,1,1]^T$
   Note that p is definitely in the triangle because this triangle is exactly those points on the defined plane in the positive octant. The calculated point is all positive meaning that it is inside the triangle.

   (c) The ray hits the sphere first, since the smallest, nonnegative t value calculated was for the sphere.

2. $\vec{\mathbf{o}} = [0,0.001,0.001]^T, \vec{\mathbf{d}} = [1,0,0]^T$:

   (a) Sphere Intersection: $a = 1$, $b = -2$, $c \approx 4.559, t = \frac{2 \pm \sqrt{-14.234}}{2}$
   Since the discriminant is negative, this ray doesn't hit the sphere.

   (b) Triangle Intersection: $a = 1$, $b = -0.998$
   $t = 0.998$ $\quad p = [0.998, 0.001, 0.001]^T$
   Note that p is definitely in the triangle because this triangle is exactly those points on the defined plane in the positive octant. The calculated point is all positive meaning that it is inside the triangle.

   (c) The ray hits the triangle first, since it doesn't hit the sphere at all and the t value for the triangle was positive.

3. $\vec{\mathbf{o}} = [0.001, 0, 0]^T, \vec{\mathbf{d}} = [0, 1, 1]^T$:

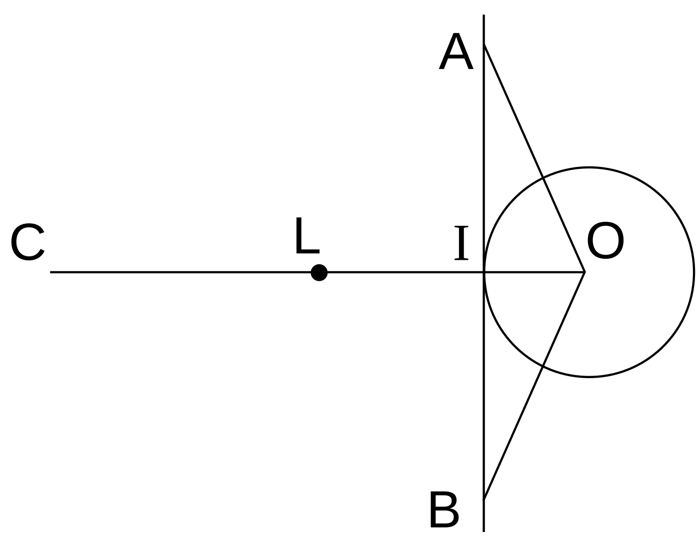
   (a) Sphere Intersection: $a = 2$, $b = -4$, $c \approx 1.436, t = \frac{4 \pm \sqrt{4.516}}{4}$
   $t_1 \approx 1.5313$ $\quad p_1 \approx [0.001, 1.5313, 1.5313]^T$
   $t_2 \approx 0.4687$ $\quad p_2 \approx [0.001, 0.4687, 0.4687]^T$

   (b) Triangle Intersection: $a = 2$, $b = -0.999$
   $t = 0.4995$ $\quad p = [0.001, 0.4995, 0.4995]^T$
   Note that p is definitely in the triangle because this triangle is exactly those points on the defined plane in the positive octant. The calculated point is all positive meaning that it is inside the triangle.

   (c) The ray hits the sphere first, since the smallest, nonnegative t value calculated was for the sphere.

Problem 3: Shading Values and Ranges For this problem, I will do all the computations for the location in the center of the image, then for the right side of the sphere. A lot of people made the mistake where they assumed the sphere was infinitely small, or that the camera was infinitely far away. As a result, we received a lot of answers that listed very simple ranges like $[-90°, 90°]$.

My pictures use C to represent the camera, L for the light, A and B for light locations, and I for the intersection point.

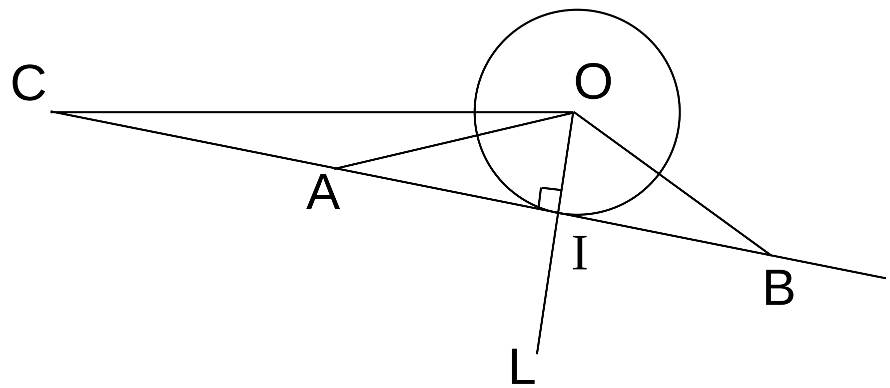


Center of the image:

In the lambertian case, we wanted to find out what the size of $\angle$COB in order to determine the non-zero range. We can see that when the light is past A or past B, then $\angle$AIC is below 0, meaning that the dot product of the light vector with the normal vector is less than 0. We know IO = 1, and AO = 3, and we can see that $\cos(\angle\text{AOC}) = \frac{\text{IO}}{\text{AO}}$. $\angle$AOC then is $\cos^{-1}\frac{1}{3} \approx 70.53$. The valid range for lambertian shading then is $0 \pm 70.53 = [-70.53°, 70.53°]$. The lambertian term is maximized when the light is aligned with the normal, at $0°$.

In the phong case, we can apply the same argument to get the same range for non-zero values. The maximum is also the same, since when the light is placed in line with the normal and camera, the half and reflected vectors are all aligned, maximizing the phong component.

Center Case	Valid Range	Maximum Value
Lambertian	$[-70.53°, 70.53°]$	$0°$
Phong	$[-70.53°, 70.53°]$	$0°$



Right side of the sphere: The first thing to notice in this case is that the light is in line with the normal not at $90°$, but at some other angle. We know that CO = 5, and IO = 1. We can also see that $\cos(\angle \mathrm{COI}) = \frac{\mathrm{IO}}{\mathrm{CO}}$. Thus, $\angle \mathrm{COI} = \cos^{-1} \frac{1}{5} \approx 78.46°$. Next, we can see that the light is perpendicular to the normal at I when in positions A and B, which are symmetric about I.

The range of valid lambertian values is $\angle \mathrm{COI} \pm \angle \mathrm{AOI}$. $\cos(\angle \mathrm{AOI}) = \frac{\mathrm{IO}}{\mathrm{AO}}$. $\angle \mathrm{AOI} = \cos^{-1} \frac{1}{3} \approx 70.53°$. The range for lambertian values then is $[7.93°, 148.99°]$. The maximum is still when the light is aligned with the normal, which would happen at $78.46°$.

For phong, it was critical to observe that the phong component is 0 whenever the light and camera are on the same side of the normal. Thus the phong component starts being non zero at $78.46°$. It continues to increase until as the light moves around the center of the sphere, until the light is occluded by the sphere at $148.99°$. The theoretical maximum for the phong component is when the reflected vector is aligned with the viewing vector, which will happen when the light is at $148.99°$.

Right Case	Valid Range	Maximum Value
Lambertian	$[7.93°, 148.99°]$	$78.46°$
Phong	$[78.46°, 148.99°]$	$148.99°$