Mappings or Functions

A mapping, or a function, is a thing that maps elements of one set to elements of another set. I'll write a function like this:

$$\begin{array}{ll} & \text{domain} \quad \text{codomain} \\ f: & \mathbb{R} \rightarrow \mathbb{R}^+ \\ : & x \mapsto x^2 \end{array}$$

← the type or signature

← the definition

this is like

```
float f(float x) {
    // return value is always ≥ 0
    return x²;
    }
```

Some sets we'll be using (more to be introduced later)

$\mathbb{R}$ – the reals

$\mathbb{R}^+$ – the non-negative reals $(\subset \mathbb{R})$

$\mathbb{Z}$ – the integers

$\mathbb{Z}^+$ – like $\mathbb{R}^+$

S^2 – the sphere

└ can think of as embedded in $\mathbb{R}^3$.

intervals

$[a, b) \in \mathbb{R}$

$[a, b] = \{x \mid a \leq x \leq b\}$

$(a, b) = \{x \mid a < x < b\}$

Cartesian products

$S \times T = \{(s,t) \mid s \in S,\ t \in T\}$ ← occasionally I'll call this a cross product but I shouldn't.

e.g. $\mathbb{R} \times \mathbb{R}$, or $\mathbb{R}^2$, is the plane.

$\mathbb{R}^3$ is 3-space

$\mathbb{R}^n$ is n-dimensional Euclidean space

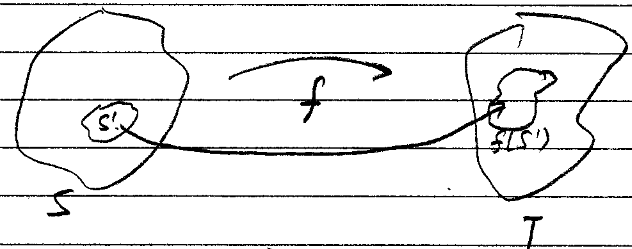
$[0,1] \times [0,1]$ unit square

Cardinality $|S|$ is the number of elements in a set

(we'll just say ∞ if S is not finite)

Inverses and properties of functions

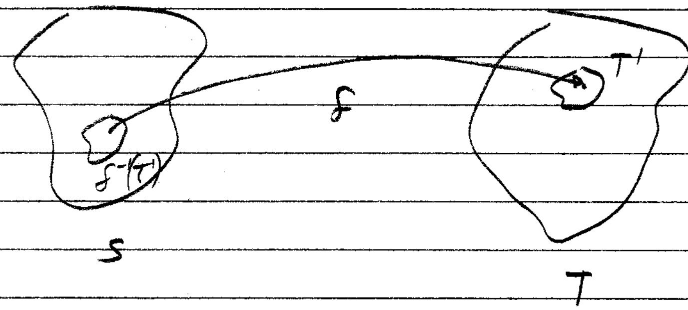
Let $f: S \to T$



some definitions.

The **image** $f(S')$ of a subset $S' \subseteq S$ is $\{f(s) \mid s \in S'\}$ or $\{t \mid (\exists s \mid f(s) = t)\}$

The **preimage** $f^{-1}(T')$ of a subset $T' \subseteq T$ is $\{s \mid f(s) \in T'\}$



The **range** of a function is the image of the domain

The fn. f is **surjective** or **onto** if $f(S) = T$

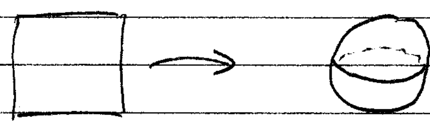
f is **injective** if $f^{-1}(t)$ is a single point for all t (that is, $|f^{-1}(t)| = 1$)

f is **bijective** if it is injective and surjective (then it establishes a 1-1 correspondence between S and T)

Iff f is **bijective** there is an inverse f^{-1} that has the property $f^{-1}(f(s)) = s$ for all $s \in S$ (and $f(f^{-1}(t)) = t \quad \forall\, t \in T$)

$\therefore (f^{-1})^{-1} = f$.

e.g. embedding a sphere in 3-space



$f: [\frac{-\pi}{2}, \frac{\pi}{2}] \times [0, 2\pi] \longrightarrow \mathbb{R}^3$

$: (\theta, \phi) \longmapsto (\cos\theta \cos\phi, \ \cos\theta \sin\phi, \ \sin\theta)$

range? the unit sphere (S^2)
injective? no!
surjective? no! (but yes if codomain is $S^2 \subseteq \mathbb{R}^3$)
inverse? no!

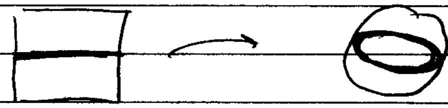
sphere embedding cont

image of $[0, 2\pi] \times \{0\}$

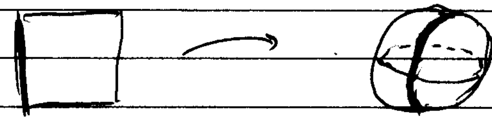
image of $\underbrace{\{0\} \times [-\frac{\pi}{2}, \frac{\pi}{2}]}$

pre-image of $f(\quad)$?

$\{0, 2\pi\} \times [-\frac{\pi}{2}, \frac{\pi}{2}] \cup [0, 2\pi] \times \{\frac{\pi}{2}, \frac{\pi}{2}\}$



the equator



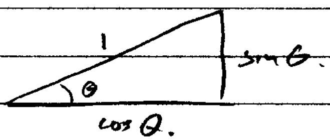
the Greenwich meridian

pre-image of x-y plane ?

pre-image of x-z plane ?

$\{0, \pi, 2\pi\} \times [-\frac{\pi}{2}, \frac{\pi}{2}] \cup [0, 2\pi] \times \{-\frac{\pi}{2}, \frac{\pi}{2}\}$

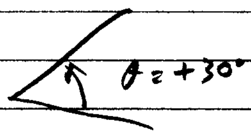
Trigonometry

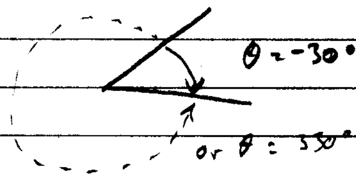


← this diagram tells us what $\sin\theta$ and $\cos\theta$ look like for $0 \leq \theta \leq 180$.

what does an angle < 0 mean?

interpretation: angles are differences in direction in $\mathbb{R}^2$.





or $\theta = 330°$ — don't care which.

inverse trig. functions

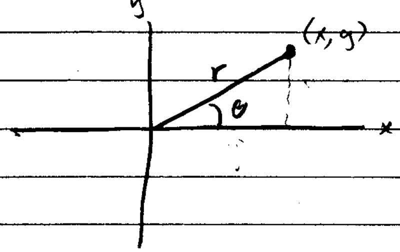
$\sin^{-1}(x)$ is the angle θ for which $\sin\theta = x$.

↑ oops. not unique! preimage of x under sin?

solution: define $\sin^{-1}(x)$ as the inverse of $\sin|_{[-\pi/2, \pi/2]}$

can do the same with $\tan^{-1}$. Problem: cartesian → polar coordinates

units: base is always radians; ° is shorthand for $\frac{\pi}{180}$



$x = r\cos\theta$

$y = r\sin\theta$

$y/x = \tan\theta$

No problem, then. $\theta = \tan^{-1}(y/x)$.

Problem: can't tell (x, y) from $(-x, -y)$

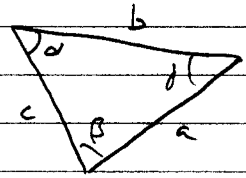
Solution: atan2 library function

if $\theta = \text{atan2}(y, x)$ then $r\cos\theta = x$ and $r\sin\theta = y$

(and $r = \sqrt{x^2+y^2}$)

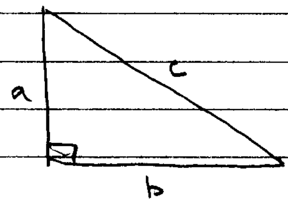
Trig identities - Shirley has handy list.

law of sines

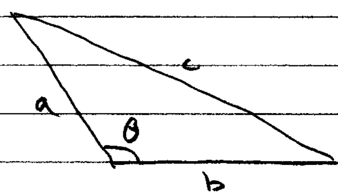


$a : \sin\alpha = b : \sin\beta = c : \sin\gamma$

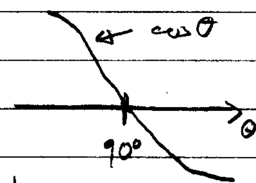
law of cosines



$c^2 = a^2 + b^2$



$c^2 = a^2 + b^2 - 2ab\cos\theta$



special cases:

$\theta = 90° \rightarrow$ Pythag.

$\theta > 90° \rightarrow \cos > 0 \rightarrow c$ longer

$\theta < 90° \rightarrow \cos < 0 \rightarrow c$ shorter

$\theta = 0 \rightarrow \cos = 1 \rightarrow (a-b)^2 = c^2$

$\theta = 180° \rightarrow \cos = -1 \rightarrow (a+b)^2 = c^2$