HW 4a Solutions

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1 2D Transformations

For each of the following transformations, transform the corners of the unit box $\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ then express the given transformation as a concatenation of multiple elementary transformations. Finally, express the transformation as an elementary transformation about a point or axis.

1.1 Transformation a

Simple matrix/vector multiplication will give us the results for transforming the corners of the unit box. Remember to add a third element to the end of each vector, and set it to 1, since these are all points. The transformed points are in the same order as they were listed above.

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

By examining the upper left 2x2 tile of the matrix, we can see this is a rotation of -90°. The right hand column indicates that a translation of $[1, 1]^T$ is applied after the rotation.

The observation that the last corner of the unit square, $[1, 0]^T$, is left in place leads to the suspicion that this is just a rotation about that point. Ater some experimentation, we can see that this is indeed the case, we are rotating -90°about $[1, 0]^T$.

1.2 Transformation b

The unit square transforms to:

$$\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix}$$

If we draw out what the transformed unit square looks like, we can see this tranformation is a shear along the x-axis by $\frac{1}{2}$, followed by a translation of $\left[\frac{1}{2}, 0\right]^T$.

Another way of looking at this, is as a shear along the line $y = -1$ by $\frac{1}{2}$.

1.3 Transformation c

The unit square transforms to:

$$\begin{bmatrix} 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Again, drawing out the resultant unit square pretty clearly shows that this transformation is a reflection across the x-axis followed by a translation of $[0, -2]^T$.

The alternate interpretation is a flip over the line $y = -1$.

1.4 Transformation d

The unit square transforms to:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{3}{2} \\ \frac{5}{2} \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} \frac{5}{2} \\ \frac{3}{2} \end{bmatrix}$$

This transformation is a little bit more complicated. Like any of the other transformations, there is more than one way to look at this. Here is my suggestion: rotate by 45°, scale by 2 along the y-axis, rotate by -45°, then finally translate by $[1, 1]^T$.

This can also be viewed as a scale by 2 against the line defined by $y + x + 2 = 0$.

2 Axis-Angle Rotations

For the next three subproblems, $\mathbf{p} = [2, 0, 3]^T$, and $\hat{\mathbf{v}} = \frac{1}{5}[3, 4, 0]^T$

2.1 Give the matrix for transforming $\mathbf{e_1}$ to $\hat{\mathbf{v}}$

This involves a simple rotation about the z-axis. This is evident because the z component of both $\mathbf{e_1}$ and $\hat{\mathbf{v}}$ are the same. The angle $\theta = \tan^{-1}\left(\frac{4}{3}\right) \approx 53.13°$

The transformation for rotating about the z-axis by θ is:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{-4}{5} & 0 & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.2 Give the transformation that takes the line segment from 0 to $0 + \mathbf{e_1}$ to the line segment from $\mathbf{p}$ to $\mathbf{p} + \hat{\mathbf{v}}$

First we must rotate the line segment by exactly the same matrix as we did in the problem above, then we must translate the line segment so that it starts at $\mathbf{p}$. Note that the rotation will not change the origin of the line segment. The translation vector will simply be that of $\mathbf{p}$ itself. The final matrix will be:

$$\begin{bmatrix} \frac{3}{5} & \frac{-4}{5} & 0 & 2 \\ \frac{4}{5} & \frac{3}{5} & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.3 Give the matrix that will rotate by 30°around the line through $\mathbf{p}$ in the direction of $\hat{\mathbf{v}}$

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First, I will spell out in english what the transformation should do, then I will write the series of matrices. Note that the order in which I do the transformations is actually reverse from the order in which I write them.

- Translate by $-\mathbf{p}$
- Rotate about the z-axis by $-\tan^{-1}\left(\frac{4}{3}\right)$
- Rotate by 30°about the x-axis
- Rotate about the z-axis by $\tan^{-1}\left(\frac{4}{3}\right)$
- Translate by $\mathbf{p}$

The first step will shift the coordinate system so that $\mathbf{p}$ is at the origin. The next step will rotate $\hat{\mathbf{v}}$ down so that it is aligned with the x-axis. The third step will do the rotation about $\hat{\mathbf{v}}$, which is now lying along the x-axis. The fourth and fifth steps will undo the first two, so that the coordinate system is back to the world coordinates.

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{-4}{5} & 0 & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{4}{5} & 0 & 0 \\ \frac{-4}{5} & \frac{3}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.4 For an arbitrary point $\mathbf{p} = [p_1, p_2, p_3]^T$ and direction $\hat{\mathbf{v}} = [v_1, v_2, v_3]^T$, write the axis-angle rotation as $MR_\theta M^{-1}$ where M is

Keep in mind throughout these next two subproblems, that M should take points in frame space *to* world space. This is counter intuitive at first glance, but when broken down, it makes sense. M^{-1} takes points from world space, and puts them in frame space. R_θ does the rotation, then M puts the points back in world space.

2.4.1 The product of a translation and two rotations about coordinate axes

This is similar to the method used in the the last part. The biggest difference is that we could possibly need two rotations and will be dealing with arbitrary values. My solution will suppose that, as above, you are moving and rotating the coordinate system so that the vector about which you are rotating becomes aligned with the x-axis. The basic out line is the same

- Rotate about the y-axis.
- Rotate about the z-axis.
- Translate by $\mathbf{p}$

The first rotation depends on v_1 and v_2. This is the only rotation we did in the last part, we just have to generalize it to any vector $\hat{\mathbf{v}}$. The angle θ_1 that we need to rotate by is easily computed using the `atan2` function. The `atan2` function takes two arguments: y and x, and computes $\tan^{-1}\left(\frac{y}{x}\right)$, but it is careful to keep the result in the correct quadrant, and handles the cases where $x = 0$. $\theta_1 = \texttt{atan2}\,(v_2, v_1)$.

The second angle, θ_2, can be computed again using `atan2`. This time, $x = \sqrt{v_1^2 + v_2^2}$ and $y = v_3$, so our equation is $\theta_2 = -\texttt{atan2}\left(v_3, \sqrt{v_1^2 + v_2^2}\right)$. The whole formula for M is:

$$\begin{bmatrix} 1 & 0 & 0 & p_1 \\ 0 & 1 & 0 & p_2 \\ 0 & 0 & 1 & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta_2) & 0 & \sin(\theta_2) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta_2) & 0 & \cos(\theta_2) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.4.2 A coordinate frame matrix constructed using cross products

Before we can construct a matrix M based on a coordinate frame about the axis and point, we need to construct the frame. The first vector will be $\hat{\mathbf{v}}$. Then we can use shirley's trick to construct a vector not aligned with $\hat{\mathbf{v}}$. Take the component of $\hat{\mathbf{v}}$ with smallest magnitude and add 1 to it. Call this vector $\mathbf{t}$. We can then construct the next two basis vectors as follows:

$$\hat{\mathbf{u}} = \frac{\mathbf{t} \times \hat{\mathbf{v}}}{\|\mathbf{t} \times \hat{\mathbf{v}}\|}, \;\; \hat{\mathbf{w}} = \hat{\mathbf{u}} \times \hat{\mathbf{v}}$$

The matrix M can then be constructed by using $\hat{\mathbf{v}}$, $\hat{\mathbf{u}}$, and $\hat{\mathbf{w}}$ as the upper left 3x3 entries of M. The upper right 3x1 entries will be $\mathbf{p}$, so that we can shift the origin to the point $\mathbf{p}$ after doing the coordinate transform. We must be careful to construct M so that $\hat{\mathbf{v}}$ is oriented along the axis about which R_θ will rotate. Lets assume this axis is the y-axis. $\hat{\mathbf{u}}$ is the x-axis then, since $\hat{\mathbf{e_1}} \times \hat{\mathbf{e_2}}$ must equal $\hat{\mathbf{e_3}}$, and $\hat{\mathbf{w}} = \hat{\mathbf{u}} \times \hat{\mathbf{v}}$. M then looks like:

$$\begin{bmatrix} \hat{\mathbf{u}} & \hat{\mathbf{v}} & \hat{\mathbf{w}} & \hat{\mathbf{p}} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

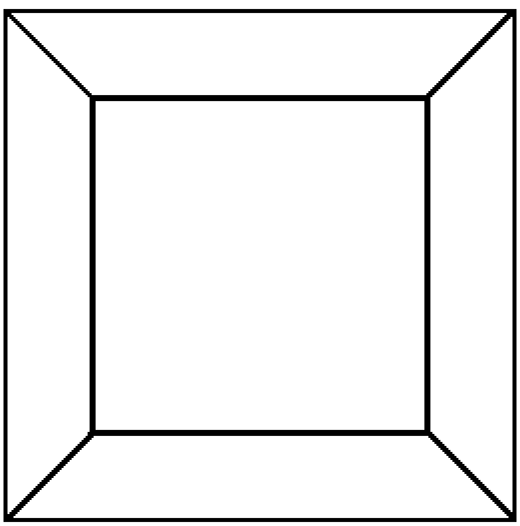
3 Viewing Transformations

3.1 What is the viewing angle of the camera?

Imagine looking at the set up from the positive x-axis. The camera is 5 units to the left of the origin, and the top of the box is 1 unit left and 1 unit up from the origin. The left edge of the box, the +z-axis, and the line from the top of the box to the camera make a right triangle. The bottom edge of the right triangle is 4 units long, and the right edge of the triangle is 1 unit long. The hypotenuse of the triangle is $\sqrt{4^2 + 1^2} = \sqrt{17}$ units long. The angle measured from the bottom edge of the triangle to the hypotenuse is half of the viewing angle. Call this angle θ. Put into equations, we have:

$$\text{viewing angle} = 2\theta,\ \tan(\theta) = \frac{1}{4} \Rightarrow \theta = \tan^{-1}\left(\frac{1}{4}\right) \Rightarrow \text{viewing angle} = 2\tan^{-1}\left(\frac{1}{4}\right) \approx 28.1^\circ$$

3.2 Sketch the image



3.3 Give the viewing and projection matrices

The viewing matrix can be constructed just as in the last part of problem 2. We need the location of the frame, and 3 basis vectors. Our basis vectors are $[1, 0, 0]^T$, $[0, 1, 0]^T$, and $[0, 0, 1]^T$, in that order. The order is important - it determines which vector is used as $\mathbf{u}$, which is $\mathbf{v}$, and which is $\mathbf{w}$. The final thing we need to do is use the location of the camera as $\mathbf{p}$. Our viewing matrix then is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The projection matrix is given in the book, all we need to do is intelligently choose the near and far plane values, as well as the top, bottom, left, and right values for the viewing volume. Since the only object in our scene is the box, we can set the near plane at the front of the box, and the far plane at the back. Since the camera is 4 units from the front of the box, $n = -4$, and $f = -6$. Also, since we are looking through the center of the window, $l = -r$, and $t = -b$. As the book mentions, the points on the near plane are left where they are, so lets pick the four remaining values by looking at the near plane. $l = -1$, and $b = -1$. This is because we are told the camera's field of view is such that the front corners of the box just intersect with the edges of the viewing volume. $r = 1$ then, and $t = 1$. Plugging these numbers in, this is our projection matrix:

$$\begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 5 & 24 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Alternately, you could use the projection matrix presented in class, with $d = 4$, giving you a 3x4 matrix of:

$$\begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

There are a number of variants on this matrix (all of which involve multiplying it by a scalar), and all are acceptable.

3.4 To where in the image will the corners project

The front corners will project to $[1,1]^T$, $[1,-1]^T$, $[-1,-1]^T$, and $[-1,1]^T$ as we start at the top right and progress clockwise. The back corners will project to $\frac{1}{3}[2,2]^T$, $\frac{1}{3}[2,-2]^T$, $\frac{1}{3}[-2,-2]^T$, and $\frac{1}{3}[-2,2]^T$ (again starting at the top right and progressing clockwise. This is easy to see — simply take the 3D coordinates of the corner in world space. Transform the point by the viewing matrix, then the projection matrix. The image coordinates of the point are specified by $x_{\text{image}} = \frac{x}{z}$, and $y_{\text{image}} = \frac{y}{z}$.

3.5 To what image coordinates will the 12 edge midpoints project?

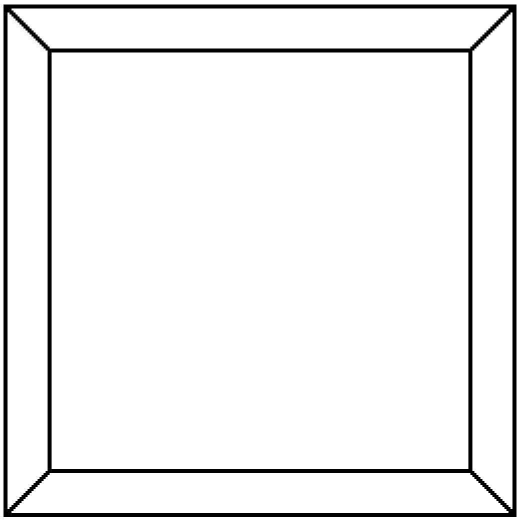
The four edges going around the back plane of the box will be very similar, as will the front four edges, and the edges going between the front and back. I will only do one of each of the groups. The t value listed below the midpoint location is the parameter along which the midpoint appears. Note that since we didn't specify an orientation, $1 - t$ is acceptable as well for a given t.

$$\begin{aligned} \text{Back Top Edge} &= \frac{1}{3}[0,2]^T \\ t &= \frac{1}{2} \\ \\ \text{Front Top Edge} &= [0,1]^T \\ t &= \frac{1}{2} \\ \\ \text{Top Right Transition Edge} &= \frac{1}{5}[4,4]^T \\ t &= \frac{2}{5} \end{aligned}$$

3.6 Do all of 3 again, but this time with the camera at $[0, 0, 10]^T$

The methods are all the same this time. I will only write down the answers for the previous parts:

Viewing Angle $= 2\cos^{-1}\left(\frac{9}{\sqrt{82}}\right) = 2\tan -1\left(\frac{1}{9}\right) \approx 12.68°$



Viewing Matrix: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Projection Matrix ($l = b = -1$, $t = r = 1$, $n = -9$, $f = -11$):

$$\begin{bmatrix} -9 & 0 & 0 & 0 \\ 0 & -9 & 0 & 0 \\ 0 & 0 & 10 & 99 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} -9 & 0 & 0 & 0 \\ 0 & -9 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Corner Projection: The front corners all project to exactly the same location. The top right back corner projects to $\frac{1}{11}[9, 9]^T$.

Edge Midpoint Projection: The front edge midpoints stay exactly the same. The top back edge midpoint projects to $\frac{1}{11}[0, 9]^T$. Both of these edge midpoints have $t = \frac{1}{2}$. The top right transition edge midpoint projects to $\frac{1}{10}[9, 9]^T$, with $t = \frac{9}{20}$. These edge midpoints are closer to being halfway...as the camera approaches an infinite distance away, these midpoint projections will approach halfway between one endpoint and the other.